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Pure Extensions, Proof Rules, and Hybrid Axiomatics

Abstract. In this paper we argue that hybrid logic is the deductive setting most natural for Kripke semantics. We do so by investigating hybrid axiomatics for a variety of systems, ranging from the basic hybrid language (a decidable system with the same complexity as orthodox propositional modal logic) to the strong Priorean language (which offers full first-order expressivity).

We show that hybrid logic offers a genuinely first-order perspective on Kripke semantics: it is possible to define base logics which extend automatically to a wide variety of frame classes and to prove completeness using the Henkin method. In the weaker languages, this requires the use of non-orthodox rules. We discuss these rules in detail and prove non-eliminability and eliminability results. We also show how another type of rule, which reflects the structure of the strong Priorean language, can be employed to give an even wider coverage of frame classes. We show that this deductive apparatus gets progressively simpler as we work our way up the expressivity hierarchy, and conclude the paper by showing that the approach transfers to first-order hybrid logic.

Keywords: Hybrid logic, modal logic, nominals, axiomatisation, completeness, proof rules.

1. Introduction

The development of Kripke semantics (initiated, among others, by Saul Kripke [27, 26, 28], Jaakko Hintikka [23], and Stig Kanger [24, 25]) was arguably the most important technical advance in the history of modal logic. Hitherto difficult questions concerning the characterisation of modal logics suddenly became simple: completeness theory reigned, and researchers such as Lemmon and Scott [29] and Segerberg [35], proved results giving elegant semantic characterisations of all the better known modal logics. True, a decade later the work of S. K. Thomason [42] and Kit Fine [19] (on the existence of frame *incomplete* logics) showed that the Kripkean paradise was more complex than anticipated, but the insights provided by work on modal completeness in the 1960s are still considered fundamental.

At the heart of Kripke semantics is a simple, but powerful, idea: view modalities as mechanisms for performing first-order quantification over worlds. Now,

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orthodox modal notation has no mechanisms for naming worlds, or asserting their equality, but when such languages are interpreted in Kripke models it becomes clear that $\Box$ and $\Diamond$ are, in essence, disguised quantifiers over worlds. This interpretation, coupled with the insight that different modal logics arise when different conditions on accessibility relations between worlds are imposed, is what makes Kripke semantics so fruitful.

This paper can be viewed as an attempt to take the Kripkean approach to its natural conclusion. We are going to be working with hybrid languages, and we are going to be examining their completeness theory in detail. What is a hybrid language? A variant of orthodox modal logic in which the notion of a world has been *internalised*. We shall examine several different kinds of hybrid languages, some very powerful, and some that are only a little stronger than orthodox modal logic. What they have in common is that they offer mechanisms for naming worlds, asserting equalities between worlds, and describing accessibility relations. Moreover, they do this in an intrinsically modal way, in essence by viewing worlds as propositions. That is, hybrid languages employ no modally-artificial devices (indeed, nowadays the hybrid languages discussed below are considered more-or-less standard by the modal logical community).

And yet the consequences of hybridisation for completeness theory are profound. It becomes straightforward to prove general completeness results, including results for frame classes that are not definable in orthodox modal languages. Moreover, the way these results are provable differs from the standard modal approach (that is, the use of canonical models). In essence, the completeness proofs we shall give are Henkin proofs. That is, our proofs make use of the standard model building machinery familiar from first-order logic — and (as will become clear) it is the extra ingredients introduced by hybridisation that makes this possible. To put it in a nutshell: hybrid languages fully internalise the concepts needed to define Kripke semantics. And this internalisation allows us to work with hybrid languages very much as if they were first-order languages. Hybrid languages thus provide a setting in which the key ideas of Kripke semantics find their clearest technical expression.

Hybrid logic has a long history. Its roots lie in philosophy. A strong form of hybrid logic was invented by Arthur Prior [32, 33] and technically explored by Robert Bull [14] in the late 1960s. The Sofia School (notably Solomon Passy and Tinko Tinchev [31] and George Gargov and Valentin Goranko [20]) proved important results about a number of different systems, including hybridised versions of Propositional Dynamic Logic. These earlier authors were well aware that hybridisation gave them access to Henkin-style model building machinery; not only do they use Henkin approaches, they also discuss the methodological shift involved ([31] is a particularly inspiring source of such discussion). But during the 1990s weaker hybrid languages were developed (notably the basic hybrid language which forms the basis of our work in this paper) and new hybrid machinery was introduced (such as the $\downarrow$ binder); it is only in the last few years that it has become clear how to handle the axiomatics of weaker hybrid languages (non-orthodox proof rules are required), what the links between Hilbert-style axiomatisations and other proof styles (such as natural deduction and tableaux) are, and how to use the Henkin machinery in a simple way.

The present paper gathers together what is known about hybrid axiomatics, generalises and in many cases simplifies it, and applies it in a systematic way all the way from the basic hybrid language to the strong languages developed by Prior. We discuss non-orthodox proof rules in detail, prove novel non-eliminability and eliminability results, and show how a new type of rule (which reflects the structure of the strong Priorean language) can be employed to give completeness for an even wider range of frame classes. We show that this deductive apparatus gets progressively simpler as we work our way up the expressivity hierarchy, and conclude the paper by showing that the approach transfers to first-order hybrid logic. In short, we put hybrid axiomatics on a general and systematic footing — a footing that is clearly first-order in nature. We view hybrid logic as the natural home of Kripke semantics, and the goal of this paper is to explain why.

2. The languages

Hybrid languages internalise worlds into modal logic. How do they do this? Arthur Prior showed the way. We add a second sort of propositional symbol to orthodox modal logic, and impose a semantic restriction on its interpretation: the new symbols must be true at *exactly one world* in any Kripke model. Such symbols are nowadays called *nominals*, and they are the heart of hybrid logic. Nominals name the unique world they are true at. To put it another way, nominals internalise worlds as propositions.

This simple change already yields a stronger logic. Consider, for example, the following orthodox modal formula:

$$\Diamond(p \wedge q) \wedge \Diamond(p \wedge r) \rightarrow \Diamond(q \wedge r).$$

This formula is *not* valid: the information that there is an accessible p and q world, and also an accessible p and r world, does not warrant the conclusion that there is an accessible q and r world. But now consider the following *hybrid* variant of the previous formula:

$$\Diamond(i \wedge q) \wedge \Diamond(i \wedge r) \rightarrow \Diamond(q \wedge r).$$

Here i is a nominal, and the substitution of i for p has resulted in a validity. For the information that there is an accessible i and q world, and also an accessible i and r world, *does* guarantee that there is an accessible q and r world. After all, i is a nominal, and nominals are true at exactly one world in any model.

Nominals are the heart of hybrid logic, but they do not work alone. The other crucial ingredient of what is nowadays called the *basic hybrid language* is the @ operator. For any nominal i and any formula ϕ we are allowed to build a new formula of the form

$$@_i\phi,$$

and such a formula is true if and only if ϕ is true at the world named by i. As we shall see when we axiomatise basic hybrid logic, @ is a useful tool. For a start, it gives us a modal theory of *world equality.* For consider the formula

$$@_ij,$$

where j is a nominal. This says that at the world named i, the nominal j is true too. That is, it says the world named by i is the same as that named by j. It is a modal analog of a first-order equality statement. Moreover, @ enables us to make *accessibility assertions.* For consider the formula

$$@_i\Diamond j.$$

This says that at the world named by i the world named j is accessible. The ability to make this style of assertion is crucial to the way our basic axiom system works. In particular, the BG rule we shall introduce (and variants such as the Paste$_\Diamond$) hinge on the use of accessibility assertions. A key theme of the paper is that hybridisation permits us to make use of essentially first-order techniques. In particular our completeness proofs are Henkin proofs, but with nominals playing the role usually played by first-order constants. It is our ability to make statements about accessibility and equality of worlds that makes Henkin machinery so straightforward to apply.

With nominals and @ introduced, we have met the two fundamental ingredients of hybrid logic. But we have not met all the devices that are nowadays considered standard in hybrid logic. Putting aside such variants as the use of the universal modality in place of @ (we shall discuss this later) let's ask ourselves a question: given that we can now name worlds using nominals, why not let ourselves quantify across them by binding nominals with $\forall$ and $\exists$? That is, why not allow ourselves to write expressions such as

$$\forall s\exists t.@_t\Diamond s$$

("For any world s, there is some world t that it is accessible from")? Why not indeed. And in fact, such languages were the earliest hybrid languages of all: Arthur Prior developed them in the late 1960s (the key sources for Prior's work on hybrid logic are [32, 33]; for a critical discussion of his contributions, see [6]). Nowadays, hybrid languages containing these binders are often called strong Priorean languages. The reason for the "strong" should be clear: they are in a completely different expressivity league from the basic hybrid language. Indeed (as Prior himself showed) they offer full first-order expressive power over worlds; more on this below.

But strong Priorean languages don't exhaust the possibilities inherent in the idea of binding nominals. Over the last decade another option has been extensively explored, namely the $\downarrow$ operator, which binds a nominal to the world of evaluation (to put is another way, $\downarrow$ allows us to create names for the current world on the fly). Here's an example, a formula that is true at a world if and only if that world has two distinct successors where p is true:

$$\downarrow s.\Diamond\downarrow t.(p \wedge @_s\Diamond\downarrow u.(p \wedge \neg t)).$$

This formula works by using $\downarrow$ to label the world of evaluation with s and then asserts that there is an accessible world (which it calls t) at which p is true. Now the interesting step: it then uses @ to assert something else about the original world s, namely that is has a successor, called u, which also makes p true, but which is distinct from t (note that $\neg t$ is true at u). The use of $\downarrow$ to store a value and of @ to retrieve and make use of that value that this formula illustrates, is typical of the way these constructs tend to be used — and is the reason for the special place $\downarrow$ occupies on the logical expressivity hierarchy.

Nominals, @, $\downarrow$ and the strong Priorean binders $\exists$ and $\forall$ are the core constructs of contemporary hybrid logic. And now that we've introduced them informally, let's make our account more precise.

The basic hybrid language: $\mathcal{H}(@)$

Given a denumerably infinite collection of propositional symbols p, q, r and so on, and a denumerably infinite collection of nominals i, j, k and so on, we define the wffs of the basic hybrid language $\mathcal{H}(@)$ over these atoms as follows:

$$\mathit{Wff} ::= i \mid p \mid \top \mid \neg\phi \mid \phi \wedge \psi \mid \Box\phi \mid @_i\phi.$$

Boolean connectives such as $\bot$, $\vee$ and $\rightarrow$ are defined in the usual way, and $\Diamond\phi$ is defined to be $\neg\Box\neg\phi$. For any nominal i, the symbol $@_i$ is called a satisfaction operator. So, syntactically speaking, the basic hybrid language is simply a multi-modal language (the modalities being $\Box$ and all the $@_i$), whose atomic symbols are subdivided into two sorts. If a formula contains no propositional variables (that is, if its only atoms are nominals) we call it a *pure* formula.

The basic hybrid language is interpreted on Kripke models. A Kripke model $\mathcal{M}$ is a triple (W, R, V), where (W, R) is a frame (that is, W is a non-empty set whose elements are called worlds, and R is a binary relation on W, the accessibility relation) and V is a valuation. But although the definition of frames is orthodox, we want nominals to act as names, so we insist that while a valuation V on a frame (W, R) is free to assign arbitrary subsets of W to ordinary propositional variables, it must assign *singleton* subsets of W to nominals. That is, we place no restrictions on the interpretation of ordinary propositional variables, but we ensure that each valuation makes every nominal true at a *unique* world. We call the unique world w in $V(i)$ the *denotation* of i under V. We interpret the basic hybrid language as follows:

$$\begin{array}{rll}
\mathcal{M}, w \models i & \text{iff} & w \in V(i) \\
\mathcal{M}, w \models p & \text{iff} & w \in V(p) \\
\mathcal{M}, w \models \top & & \text{always} \\
\mathcal{M}, w \models \neg\phi & \text{iff} & \text{not } \mathcal{M}, w \models \phi \\
\mathcal{M}, w \models \phi \wedge \psi & \text{iff} & \mathcal{M}, w \models \phi \ \text{ and } \ \mathcal{M}, w \models \psi \\
\mathcal{M}, w \models \Box\phi & \text{iff} & \text{for all } v \in W \text{ such that } Rwv,\ \mathcal{M}, v \models \phi \\
\mathcal{M}, w \models @_i\phi & \text{iff} & \mathcal{M}, d \models \phi. \text{ where } d \text{ is the denotation of } i \text{ under } V.
\end{array}$$

Save for the clauses for nominals and @, these are the familiar Kripke-style interpretation clauses for orthodox propositional modal logic. As usual, $\mathcal{M} \models \phi$ means that ϕ is true at all worlds in $\mathcal{M}$, $\mathcal{F} \models \phi$ means that ϕ is valid on the frame $\mathcal{F}$ (that is, that ϕ is true at all worlds in $\mathcal{F}$ regardless of the valuation that is used), and $\models \phi$ means that ϕ is valid on all frames.

It should be clear that $\mathcal{H}(@)$ is a relatively simple extension of orthodox modal logic. This is borne out by the following result: the problem of determining whether a basic hybrid formula is satisfiable is *decidable*. Moreover, the complexity of this problem is (up to a polynomial) no more complex than the problem for orthodox modal logic. For we have:

THEOREM 1. *The satisfiability problem for the basic hybrid logic is* PSPACE-*complete.*

PROOF. See [1]. ∎

But in spite of its simplicity, the basic hybrid language is surprisingly strong when it comes to defining properties of frames. For a start, many of the properties definable in the basic modal language can be defined using pure formulas.

As the reader can easily check, for each of the properties listed below, a frame $\mathcal{F}$ has the property stated if and only if it validates the hybrid formula listed alongside:

$$\begin{array}{rll} \forall w Rww & \textit{reflexivity} & @_i\Diamond i \\ \forall wu(Rwu \to Ruw) & \textit{symmetry} & @_i\Box\Diamond i \\ \forall wuv(Rwu \wedge Ruv \to Rwv) & \textit{transitivity} & @_i\Diamond j \wedge @_j\Diamond k \to @_i\Diamond k \\ \forall wu(Rwu \to \exists v(Rwv \wedge Rvu)) & \textit{density} & \Diamond i \to \Diamond\Diamond i \end{array}$$

Moreover, pure formulas also enable us to define many properties that are *not* definable in the basic modal language. Here are some well-known examples:

$$\begin{array}{rll} \forall w \neg Rww & \textit{irreflexivity} & @_i\neg\Diamond i \\ \forall wu(Rwu \to \neg Ruw) & \textit{asymmetry} & @_i\neg\Diamond\Diamond i \\ \forall wu(Rwu \wedge Ruw \to u = w) & \textit{antisymmetry} & i \to \Box(\Diamond i \to i) \\ \forall wu(Rwu \vee w = u \vee Ruw) & \textit{trichotomy} & @_j\Diamond i \vee @_j i \vee @_i\Diamond j \end{array}$$

All the frame properties defined above are first-order. This is no coincidence: all pure formulas define first-order frame conditions (see [9, 36, 3]). But not only do pure formulas define first-order properties, when used as axioms they are also automatically *complete* with respect to the class of frames they define (see [31, 20, 7]). More precisely, it is possible to define a base logic that is complete with respect to the class of all frames, and that extends automatically to a complete system for the class of frames defined by any set of pure axioms.

These two simple observations motivate much of the work that follows (and indeed, other papers on hybrid axiomatics stretching all the way back to the work of Bull). Exploiting these observations (in the basic hybrid language) turns out to require the use of non-orthodox proof rules, rules which let us build Henkin models. And extending these ideas to classes of frames not covered by pure formulas also leads to non-orthodox rules, rules inspired by the strong Priorean language.

For now, that's all we need to know about $\mathcal{H}(@)$, but before moving on and defining the $\downarrow$ binder, let's briefly consider $\mathcal{H}(A)$, an alternative to $\mathcal{H}(@)$ that uses the universal modality (or the global modality, as it is sometimes called) instead of @. The diamond form of the universal modality is written E, the box form is written A, and they have the following semantics: $E\phi$ means ϕ is true at *some* world in the model, and $A\phi$ means ϕ is true at *all* worlds in the model. That is:

$$\begin{array}{lll} \mathcal{M}, w \models A\phi & \text{iff} & \text{for all } v \in W,\ \mathcal{M}, v \models \phi \\ \mathcal{M}, w \models E\phi & \text{iff} & \text{for some } v \in W,\ \mathcal{M}, v \models \phi. \end{array}$$

The universal modality is interesting for a number of reasons (it is often useful to be able to impose global constraints on models) but for present purposes its most important property is that it is strong enough to define the @ operator. For

$$E(i \wedge \phi)$$

says that "somewhere in the model there is a world where i is true, and at that world ϕ is true too", and this is exactly what $@_i\phi$ means. That is, we can define

$$@_i\phi =_{def} E(i \wedge \phi).$$

Alternatively, we can define

$$@_i\phi =_{def} A(i \rightarrow \phi).$$

This says "at all worlds in the model where i is true, ϕ is true too", and this has the same effect.

Thus $\mathcal{H}(A)$ is at least as expressive as $\mathcal{H}(@)$. And it is not difficult to show that it actually strictly *more* expressive: we cannot define the universal modality in terms of @. This expressive difference is reflected in the computational behaviour of the two languages. We have already remarked that $\mathcal{H}(@)$ is decidable, and indeed $\mathcal{H}(A)$ is too. However, whereas $\mathcal{H}(@)$ has a PSPACE-complete satisfiability problem, the satisfiability problem for $\mathcal{H}(A)$ is known to be EXPTIME-complete (see [22]). Assuming that the standard assumptions about complexity classes are correct, this means $\mathcal{H}(A)$ has a harder satisfiability problem. As we shall see, however, deductively the two languages are not so different: axiomatising $\mathcal{H}(A)$ turns out to be extremely straightforward. Indeed, we will be able to do so by adding one single axiom (namely Ei) to our axiomatisation of $\mathcal{H}(@)$.

The basic language with downarrow: $\mathcal{H}(@, \downarrow)$

Let's now be precise about the syntax and semantics of the $\downarrow$ binder. When we explained it informally, we said that $\downarrow$ allowed us to "bind a nominal" to the world of evaluation. Now, it would be possible to define its binding syntax in this way, but we think it is somewhat cleaner to draw a notational distinction between bindable and non-bindable symbols. Accordingly, we shall assume that we have a denumerably infinite set of *world variables* (typically written s, t, u, and so on) at our disposal that are distinct from both nominals and orthodox propositional variables. Syntactically and semantically world variables will behave just like nominals, except that they will be open to binding by $\downarrow$, and nominals will not. Accordingly, we define the syntax of $\mathcal{H}(@, \downarrow)$, the language obtained by adding world variables and $\downarrow$ to $\mathcal{H}(@)$, as follows:

$$Wff ::= s \mid i \mid p \mid \top \mid \neg\phi \mid \phi \wedge \psi \mid \Box\phi \mid @_i\phi \mid @_s\phi \mid \downarrow s.\phi$$

Free and bound world variables are defined in the obvious way, as are other syntactic concepts (such as the scope of an occurrence of $\downarrow$). A formula of $\mathcal{H}(@, \downarrow)$ that contains no free variables is called a sentence. Note that we allow @ to make use of world variables: this gives rise to the "store and retrieve" interplay between $\downarrow$ and @.

Now for the semantics. As with $\mathcal{H}(@)$, we interpret the language on Kripke models, but we now need a mechanism for coping with free world variables. But this is standard: we simply introduce assignments of values to world variables, and relativise the evaluation of formulas to a variable assignment. So, given a Kripke model $\mathcal{M} = (W, R, V)$, an assignment of values to variables g on $\mathcal{M}$ is a function from the set of world variables to W. Furthermore, if g and g' are assignments of values to variables on $\mathcal{M}$, and g' differs from g, if at all, only in what it assigns to s, then we say that g' is an s-variant of g, and in such a case we write $g' \overset{s}{\sim} g$. We can now give the satisfaction definition:

$$
\begin{array}{rcl}
\mathcal{M}, g, w \models s & \text{iff} & w = g(s) \\
\mathcal{M}, g, w \models i & \text{iff} & w \in V(i) \\
\mathcal{M}, g, w \models p & \text{iff} & w \in V(p) \\
\mathcal{M}, g, w \models \top & & \text{always} \\
\mathcal{M}, g, w \models \neg\phi & \text{iff} & \text{not } \mathcal{M}, g, w \models \phi \\
\mathcal{M}, g, w \models \phi \wedge \psi & \text{iff} & \mathcal{M}, g, w \models \phi \text{ and } \mathcal{M}, g, w \models \psi \\
\mathcal{M}, g, w \models \Box\phi & \text{iff} & \text{for all } v \in W \text{ such that } Rwv,\ \mathcal{M}, g, v \models \phi \\
\mathcal{M}, g, w \models @_i\phi & \text{iff} & \mathcal{M}, g, d \models \phi. \text{ where } d \text{ is the denotation of } i \text{ under } V \\
\mathcal{M}, g, w \models @_s\phi & \text{iff} & \mathcal{M}, g, d \models \phi. \text{ where } d \text{ is } g(s) \\
\mathcal{M}, g, w \models \ \downarrow s.\phi & \text{iff} & \mathcal{M}, g', w \models \phi, \text{where } g' \overset{s}{\sim} g \text{ and } g(s) = w.
\end{array}
$$

The language $\mathcal{H}(@, \downarrow)$ has attracted a great deal of interest in recent years, for there are a number of results which shows that it occupies an interesting niche in the logical expressivity hierarchy. We shall note some of these results when we axiomatise the language, but one is worth mentioning right away. As was shown in [1], the language $\mathcal{H}(@, \downarrow)$ is expressively equivalent with the *bounded fragment* of first-order logic. The bounded fragment consists of all first-order formulas built up from atomic formulas using the booleans and bounded quantifications of the form $\exists y(R\tau y \wedge \phi)$ and $\forall y(R\tau y \rightarrow \phi)$, where τ is a term that does not contain y, and R is interpreted as the accessibility relation. The bounded fragment arises naturally in set theory (see [30]) and arithmetic (see [15]). In the mid-1960s, Feferman and Kreisel [18, 17] characterised the bounded fragment as the fragment of first-order logic invariant under generated submodels. Hybrid logicians invented $\downarrow$ because of the elegant "store and retrieve" interplay that $\downarrow$ and @ exhibit; it is intriguing that two such different lines of investigation should have led to essentially the same logic.

The strong Priorean language: $\mathcal{H}(@,\forall)$

Let us now define the most expressive hybrid language considered in this paper, the strong Priorean language $\mathcal{H}(@,\forall)$. As with $\mathcal{H}(@,\downarrow)$, this language enables us to "bind nominals", and as before we implement this (syntactically) by making use of world variables, and (semantically) by making use of assignments of values to variables. So assume the set of world variables has been fixed. Then the syntax of the strong Priorean language is:

$$\mathit{Wff} ::= s \mid i \mid p \mid \top \mid \neg\phi \mid \phi \wedge \psi \mid \Box\phi \mid @_i\phi \mid @_s\phi \mid \forall s.\phi$$

As with $\mathcal{H}(@,\downarrow)$, free and bound world variables are defined in the obvious way, and a formula is called a sentence if it contains no free variables. We define $\exists s.\phi$ to be $\neg\forall s.\neg\phi$.

The semantics contains no surprises either. We use assignments of values to variables in the standard way to define

$$\mathcal{M}, g, w \models \forall s.\phi \quad \text{iff} \quad \mathcal{M}, g', w \models \phi, \text{for all } g' \stackrel{s}{\sim} g.$$

It follows that

$$\mathcal{M}, g, w \models \exists s.\phi \quad \text{iff} \quad \mathcal{M}, g', w \models \phi, \text{for some } g' \stackrel{s}{\sim} g.$$

That is, $\forall$ and $\exists$ are duals, just as we would expect.

Clearly $\mathcal{H}(@,\forall)$ is an expressive language. It is at least as expressive as $\mathcal{H}(@,\downarrow)$, as $\downarrow s.\phi$ is simply $\exists s.(s \wedge \phi)$. But in fact $\mathcal{H}(@,\forall)$ is strictly more expressive than $\mathcal{H}(@,\downarrow)$. We have already remarked that $\mathcal{H}(@,\downarrow)$ is known to be expressively equivalent with the bounded fragment of first-order logic. However $\mathcal{H}(@,\forall)$ is stronger than this: *any* first-order expression in a language with a binary relation R (for talking about accessibility) and unary relations P (for talking about propositional information) can be translated into $\mathcal{H}(@,\forall)$. Here's how to do it (incidentally, the following translation, which is nowadays called the *hybrid translation*, was known to Arthur Prior in the late 1960s):

$$\begin{array}{lcl}
\text{HT}(Rst) & = & @_s\Diamond t \\
\text{HT}(Ps) & = & @_s p \\
\text{HT}(s = t) & = & @_s t \\
\text{HT}(\neg\phi) & = & \neg\text{HT}(\phi) \\
\text{HT}(\phi \wedge \psi) & = & \text{HT}(\phi) \wedge \text{HT}(\psi) \\
\text{HT}(\exists v.\phi) & = & \exists v.\text{HT}(\phi) \\
\text{HT}(\forall v.\phi) & = & \forall v.\text{HT}(\phi).
\end{array}$$

Note the use of @ for handling the translation of atomic formulas. These uses of @ are crucial: if we don't have @ in the language (either as a primitive, or defined using some stronger modality, such as the universal modality) then it is *not* possible to translate all first-order formulas. To put it another way, in the absence of @, the ability to bind nominals with $\exists$ and $\forall$ does *not* give rise to full first-order expressive power! This (at first glance counterintuitive) result is proved in [9]. Roughly speaking, in the strong hybrid language, classical quantification is factored into a "binding" step (the task of $\exists$ and $\forall$) and a "carry out the evaluation step *there*" step (which is based on @). These two functions, which are conflated in first-order logic, are teased apart in hybrid logic.

Indeed, this teasing apart of variable binding and evaluation can be carried even further, for it turns out that $\mathcal{H}(A, \downarrow)$ (that is, $\mathcal{H}(@, \downarrow)$ enriched with the universal modality) is expressively equivalent to $\mathcal{H}(@, \forall)$. To see this, note that $\forall s.\phi$ is equivalent to

$$\downarrow t.A\downarrow s.@_t\phi, \text{ where } t \text{ does not occur in } \phi.$$

Thus $\mathcal{H}(A, \downarrow)$ can define the Priorean binders, and we have already noted that A enables us to define @.

Finally, note that $\mathcal{H}(@, \forall)$ and $\mathcal{H}(A, \forall)$ are expressively equivalent. That $\mathcal{H}(A, \forall)$ is at least as expressive as $\mathcal{H}(@, \forall)$ is clear. But $\mathcal{H}(@, \forall)$ is strong enough to define the universal modality, for $A\phi$ can be viewed as shorthand for $\forall s.@_s\phi$, where s does not occur in ϕ.

3. An axiomatisation of $\mathcal{H}(@)$

We begin by presenting an axiomatisation of the weakest language, the basic hybrid language $\mathcal{H}(@)$. Recall that this language contains no binders, and that over the class of all frames its satisfiability problem is decidable, indeed PSPACE-complete. Although expressively weak compared with $\mathcal{H}(@, \downarrow)$ or the strong Priorean language $\mathcal{H}(@, \forall)$, we shall now show that $\mathcal{H}(@)$ is deductively strong. In particular, axioms and rules that make it possible to carry out a Henkin-style model construction are already stateable in this modest extension of orthodox modal logic. Moreover, lifting this axiomatisation to stronger hybrid languages is straightforward.

The axiomatisation in Figure 1 is a simplification of the one given in [7] (which was in turn based on the axiomatization in [12]). A remark on the substitution rule: here, σ is any substitution that uniformly replaces nominals by nominals and atomic propositions by arbitrary formulas. In hybrid logic,

$\mathbf{K}_{\mathcal{H}(@)}$	
Axioms:	
CT	All classical tautologies
$\mathrm{K}_{\Box}$	$\vdash \Box(p \to q) \to (\Box p \to \Box q)$
$\mathrm{K}_{@}$	$\vdash @_i(p \to q) \to (@_i p \to @_i q)$
$\mathrm{Selfdual}_{@}$	$\vdash @_i p \leftrightarrow \neg @_i \neg p$
$\mathrm{Ref}_{@}$	$\vdash @_i i$
Agree	$\vdash @_i @_j p \leftrightarrow @_j p$
Intro	$\vdash i \to (p \leftrightarrow @_i p)$
Back	$\vdash \Diamond @_i \phi \to @_i \phi$
Rules:	
MP	If $\vdash \phi$ and $\vdash \phi \to \psi$ then $\vdash \psi$
Subst	If $\vdash \phi$ then $\vdash \phi^{\sigma}$
$\mathrm{Gen}_{@}$	If $\vdash \phi$ then $\vdash @_i \phi$
$\mathrm{Gen}_{\Box}$	If $\vdash \phi$ then $\vdash \Box\phi$
Name	If $\vdash @_i \phi$ and i does not occur in ϕ, then $\vdash \phi$
BG	If $\vdash @_i \Diamond j \to @_j \phi$ and $j \neq i$ does not occur in ϕ, then $\vdash @_i \Box \phi$

Figure 1. The $\mathcal{H}(@)$ axiomatisation.

the atomic symbols of our languages are sorted, and the substitution rule must respect this.

The axiomatization is sound and complete with respect to the class of all frames, and moreover completeness is guaranteed for arbitrary extensions with *pure* axioms. Recall that a formula is pure if it contains no ordinary propositional variables; for $\mathcal{H}(@)$ this means that the only atomic symbols a pure formula can contain are nominals. For any set of pure $\mathcal{H}(@)$ formulas Λ, let $\mathbf{K}_{\mathcal{H}(@)} + \Lambda$ denote the above axiomatization extended with the axioms in Λ. Then we have:

THEOREM 2 (Completeness). *Let Λ be any set of pure $\mathcal{H}(@)$ axioms. A set of $\mathcal{H}(@)$ formulas Σ is $\mathbf{K}_{\mathcal{H}(@)} + \Lambda$ consistent iff Σ is satisfiable in a model based on a frame satisfying the frame properties defined by Λ.*

Let's consider some concrete examples of what this theorem tells us. First, if we choose Λ to be empty set (that is, if we add no additional axioms) then it says that $\mathbf{K}_{\mathcal{H}(@)}$ is complete with respect to the class of all frames. On the other hand, if we choose Λ to be $\{@_i \Diamond j \wedge @_j \Diamond k \to @_i \Diamond k\}$, then we have an axiomatic system that is complete with respect to the class of transitive frames.

And if we choose

$$\Lambda = \{@_i \neg \Diamond i, @_i \Diamond j \wedge @_j \Diamond k \to @_i \Diamond\},$$

then we have an axiomatisation that is complete with respect to strict partial orders, for the first axiom defines irreflexivity, and the second transitivity. In short, the theorem is reminiscent of the completeness result for first-order logic. In first-order logic we have a basic axiom system, and when this axiom system is extended with additional first-order axioms Λ, the resulting axiomatisation is complete with respect to the class of models for Λ. Now, the theorem for $\mathbf{K}_{\mathcal{H}(@)}$ is not as general as this (completeness is only guaranteed for pure axiomatic extensions, not arbitrary extensions). Nonetheless its simplicity contrasts strongly with the situation in orthodox modal logic. General completeness results are known for orthodox modal languages; Sahlqvist's Completeness Theorem [34] is probably the best known. But such results typically place relatively complex syntactic constraints on the permissible forms of axioms; in hybrid logic we need merely demand purity.

Theorem 2 is essentially the result proved in [12, 7], though the axiom system used here is simpler. Our first goal is to prove a generalisation of this theorem that covers more frame classes. We will do so this by introducing additional rules, rules which allow us to cope with frame classes for which there is no pure axiom. And this brings us to one of the main themes of the paper: the status of non-orthodox proof rules in hybrid logic. So, before going further and introducing even more rules, let us discuss the non-orthodox proof rules already present in our axiomatisation for $\mathcal{H}(@)$, namely the Name and BG rules. What do these rules say, and what do they do for us?

Let's start with the BG rule. This stands for *Bounded Generalisation*, and as the name is meant to suggest, it is a modal analog of the UG (Universal Generalisation) rule of first-order logic. Let's look carefully at what it says:

If $\vdash @_i \Diamond j \to @_j \phi$ and $j \neq i$ does not occur in ϕ, then $\vdash @_i \Box \phi$.

Because j is a nominal distinct from i that does not occur in ϕ, we can read $@_i \Diamond j$ as asserting the existence of an world (arbitrarily labelled j) accessible from the world labelled i. Accordingly, the rule can be read as follows: suppose we can prove that for an arbitrary world j accessible from i, that ϕ holds at j. Then, since j is arbitrary, ϕ must hold at *all* worlds accessible from i (that is, $@_i \Box \phi$). The analogy with the first-order rule of universal generalisation should be clear. The only real difference is that whereas the first-order quantifiers are global in their reach, the $\Diamond$ and $\Box$ operators only control what happens at accessible worlds, hence the quantificational force of the BG rule is 'bounded'.

It's also worth noting that our informal explanation of the BG rule has a natural deduction flavour, and in fact the $\Box$-introduction rule in the natural deduction system for hybrid logic given in [13] is closely related to BG.

What of the Name rule? This tells us that if it is provable that ϕ holds at an arbitrary world i (the world is arbitrary because i does not occur in ϕ) then we can prove ϕ. This rule plays a simple, but crucial, role in the completeness proof given below. Moreover, it has a clear motivation in terms of hybrid tableaux. In what are called 'internalised' hybrid tableaux systems (for example, the systems in [5]), to prove ϕ we attempt to falsify ϕ at some arbitrary world i. That is, to prove ϕ we actually apply the tableaux rules to $\neg @_i \phi$ (where i is some nominal not occurring in ϕ) rather than to ϕ itself. The Name rule can be thought of as an the axiomatic equivalent of this procedure. Indeed, there is an interesting formal similarity. Just as the labelling with an arbitrary world i takes place only once in a tableaux proof (namely, at the very first step) we can show that the Name rule is only ever needed once in an axiomatic proof, namely at the very last step. The proof is given in Figure 2. The entries in the table show how to systematically permute applications of the Name rule downwards with respect to other rule applications, and how to 'cancel' adjacent applications of Name, so that we end up with a single application of Name at the root of the proof tree.

We shall discuss the role of Name and BG in more depth in the following section (indeed, we shall show that the use of non-orthodox rules is indispensable to the proof of Theorem 2). But for now, let's simply accept them, and turn to the task of generalising Theorem 2. As we have already remarked, Theorem 2 is a simple and general result, rather in the style of the completeness theorem for first-order logic. Moreover it covers frame conditions that are undefinable in orthodox modal logic; irreflexivity is an example of such a condition. Nonetheless, there are also many interesting frame classes that are not definable using pure $\mathcal{H}(@)$ axioms. One example is the class of frames in which every world has a predecessor, that is, the class of frames satisfying the $\mathcal{H}(@, \forall)$ sentence $\forall s \exists t @_t \Diamond s$. Another, perhaps more appealing example is the class of Church-Rosser frames, that is, frames satisfying the $\mathcal{H}(@, \forall)$ sentence

$$\forall stu \exists v (@_s \Diamond t \wedge @_s \Diamond u \rightarrow @_t \Diamond v \wedge @_u \Diamond v).$$

For proofs that these conditions are not definable in $\mathcal{H}(@)$, see [36, 3].

But the fact that both conditions have straightforward (namely, $\forall\exists$-prenex) formulations in $\mathcal{H}(@, \forall)$ should give us pause for thought. It is true that we can't use them as axioms (they don't belong $\mathcal{H}(@)$) but can't we make use of them in some other way? It turns out that we can. As we shall now show,

MP_l	$\dfrac{\dfrac{@_i\phi}{\phi}\,\textbf{(Name)} \qquad \phi \to \psi}{\psi}$	$\leadsto$	$\dfrac{@_i\phi \qquad \dfrac{\phi \to \psi}{@_i\phi \to @_i\psi}\,(Gen_@, K_@)}{\dfrac{@_i\psi}{\psi}\,\textbf{(Name)}}$
MP_r	$\dfrac{\phi \qquad \dfrac{@_i(\phi \to \psi)}{\phi \to \psi}\,\textbf{(Name)}}{\psi}$	$\leadsto$	$\dfrac{\dfrac{\phi}{@_i\phi}\,(Gen_@) \qquad \dfrac{@_i(\phi \to \psi)}{@_i\phi \to @_i\psi}\,(K)}{\dfrac{@_i\psi}{\psi}\,\textbf{(Name)}}$
$\mathrm{Gen}_@$	$\dfrac{\dfrac{@_i\phi}{\phi}\,\textbf{(Name)}}{@_j\phi}\,(Gen_@)$	$\leadsto$	$\dfrac{@_i\phi}{@_j\phi}\,(Subst)$
Subst	$\dfrac{\dfrac{@_i\phi}{\phi}\,\textbf{(Name)}}{\phi^\sigma}\,(Subst)$	$\leadsto$	$\dfrac{\dfrac{@_i\phi}{@_i\phi^\sigma}\,(Subst)}{\phi^\sigma}\,\textbf{(Name)}$
Name	$\dfrac{\dfrac{@_i@_j\phi}{@_j\phi}\,\textbf{(Name)}}{\phi}\,\textbf{(Name)}$	$\leadsto$	$\dfrac{\dfrac{@_i@_j\phi}{@_j\phi}\,(Agree)}{\phi}\,\textbf{(Name)}$
BG	$\dfrac{\dfrac{@_k(@_i\Diamond j \to @_j\phi)}{@_i\Diamond j \to @_j\phi}\,\textbf{(Name)}}{@i\Box\phi}\,(BG)$	$\leadsto$	$\dfrac{\dfrac{@_k(@_i\Diamond j \to @_j\phi)}{@_i\Diamond j \to @_j\phi}\,(K_@, Agree)}{@_i\Box\phi}\,(BG)$

Figure 2. The Name rule is needed only once, at the very end of the proof.

$\forall\exists$-prenex sentences of $\mathcal{H}(@,\forall)$ give rise to what we call *existential saturation rules.*

Let ξ be an $\mathcal{H}(@,\forall)$ sentence $\forall s_1 \dots s_n \exists t_1 \dots t_m.\phi(s_1, \dots, s_n, t_1, \dots, t_m)$, where ϕ is quantifier-free, pure, and nominal free. That is, the only atomic symbols in ξ are bound world variables. Then ξ gives rise to the following proof rule:

> If $\vdash \phi(i_1, \dots, i_n, j_1, \dots, j_m) \to \psi$ then $\vdash \psi$, where $i_1, \dots, i_n, j_1, \dots, j_m$ are distinct, and $j_1, \dots, j_m$ do not occur in ψ.

For example, the rule corresponding to $\forall stu\exists v(@_s\Diamond t \wedge @_s\Diamond u \to @_t\Diamond v \wedge @_u\Diamond v)$ (the $\mathcal{H}(@,\forall)$ sentence expressing the Church-Rosser property) is:

If $\vdash (@_i\Diamond j \wedge @_i\Diamond k \rightarrow @_j\Diamond l \wedge @_k\Diamond l) \rightarrow \psi$ then $\vdash \psi$,
where l does not occur in ψ and is distinct from i, j, k.

In short, we make an existential saturation rule out of a $\forall\exists$-prenex $\mathcal{H}(@,\forall)$ formula by using nominals to skolemise the bound world variables away (which yields a $\mathcal{H}(@)$ formula). We are then free to use the resulting skolem form in proofs in the manner the rule states. Before going any further, let's check that such rules are sound. We say that a frame $\mathfrak{F}$ *admits* an existential saturation rule if whenever $\mathfrak{F} \models \phi(i_1, \ldots, i_n, j_1, \ldots, j_m) \rightarrow \psi$ then $\mathfrak{F} \models \psi$ too. Then we have:

LEMMA 1. *Let ρ be an existential saturation rule for some $\mathcal{H}(@,\forall)$ sentence of the form $\forall s_1 \ldots s_n \exists t_1 \ldots t_n.\phi$, where ϕ is quantifier-free, pure, and nominal free. Then every frame satisfying ξ admits ρ as a rule of proof.*

PROOF. Suppose $\mathfrak{F} \models \forall s_1 \ldots s_n \exists t_1 \ldots t_n.\phi$, and suppose that the antecedent of ρ is valid on $\mathfrak{F}$. That is, suppose $\mathfrak{F} \models \phi(i_1, \ldots, i_n, j_1, \ldots, j_m) \rightarrow \psi(i_1, \ldots, i_n)$, where $i_1, \ldots, i_n, j_1, \ldots, j_m$ are distinct, and $j_1, \ldots, j_m$ do not occur in ψ. We want to show that $\mathfrak{F} \models \psi(i_1, \ldots, i_n)$. But this is straightforward. Pick any world w and any valuation V. Since $\mathfrak{F} \models \forall s_1 \ldots s_n \exists t_1 \ldots, t_n.\phi$, there is a valuation V' which agrees with V on the values it assigns to $i_1, \ldots, i_n$, and which assigns worlds to $j_1, \ldots, j_m$ in such a way that ϕ is satisfied at w under V'. But then ψ is also satisfied at w under V'. But as ψ contains no occurrences of $j_1, \ldots, j_m$, it follows that ψ is satisfied at w under V too. As V and w were arbitrary, $\mathfrak{F} \models \psi(i_1, \ldots, i_n)$ as required. ■

In short, if we want to axiomatize frame classes involving properties such as the Church-Rosser property, we can add the relevant existential saturation rules to the axiomatization without losing soundness. Indeed, as we shall soon see, the addition of such rules guarantees completeness with respect to the relevant frame class.

It's also worth remarking that existential saturation rules generalise the idea underlying the use of pure axioms. The $\mathcal{H}(@)$ axiom for irreflexivity is $@_i\neg\Diamond i$. But this can be viewed as arising by skolemisation from the $\forall s.@_s\neg\Diamond s$, a $\forall$-prenex $\mathcal{H}(@,\forall)$ formula. And because the quantification here is purely universal, we *don't* need a rule to capture its effect when we skolemise: we simply use $@_i\neg\Diamond i$ as an axiom. But this is the only difference. The conditional involved in the statement of the existential saturation rules ("If $\vdash \phi(i_1, \ldots, i_n, j_1, \ldots, j_m) \rightarrow \psi$ then $\vdash \psi$") is merely the obvious way of extending to $\forall\exists$-prenex formulas the idea underlying the use of pure formulas. It would be interesting to further generalise the basic idea, that is, to try and capture the effect of $\mathcal{H}(@,\forall)$ formulas with more complex prenex forms, but we won't pursue that line of enquiry here.

One other remark is worth making. Existential saturation rules are natural in the setting of tableaux proof systems; there they are rules which license the creation of new tableaux nodes. Indeed it was our work on tableaux system for frame classes not definable by pure formulas that first led us to investigate them. For further details, see [10].

We shall now state and prove a generalisation of Theorem 2. First two remarks. There is a clear correspondence between existential saturation rules and $\forall\exists$-prenex $\mathcal{H}(@\forall)$ sentences, so in what follows we sometimes talk as if they were interchangeable; in particular, when we talk about the frame class defined by an existential saturation rule (as we do in the statement of the following theorem) we simply mean the class of frames defined by the corresponding $\mathcal{H}(@,\forall)$ sentence. Secondly, note that in countable languages, there are only countably many existential saturation rules.

Given a set of pure axioms Λ and a set of existential saturation rules P, we will use $\mathbf{K}_{\mathcal{H}(@)} + \Lambda + P$ to denote the $\mathbf{K}_{\mathcal{H}(@)}$ axiomatization extended with the axioms in Λ and the rules in P. Here is the theorem we shall prove:

THEOREM 3 (Completeness with existential saturation rules). *Let Λ be a set of pure $\mathcal{H}(@)$ axioms and let P be a set of existential saturation rules. A set of $\mathcal{H}(@)$ formulas Σ is $\mathbf{K}_{\mathcal{H}(@)} + \Lambda + P$ consistent iff Σ is satisfiable in a model satisfying the frame properties defined by Λ and P.*

The remainder of this section is dedicated to the proof of Theorem 3. Its structure closely resembles that of a Henkin-style completeness proof for first-order logic, with nominals playing the role of first-order constants. We begin by showing the derivability of a number of useful validities and rules.

LEMMA 2. *The following are derivable*

$K_@^{-1}$	$\vdash (@_i\phi \to @_i\psi) \to @_i(\phi \to \psi)$
Nom	$\vdash @_ij \to (@_i\phi \leftrightarrow @_j\phi)$
Sym	$\vdash @_ij \to @_ji$
Bridge	$\vdash @_i\Diamond j \wedge @_j\phi \to @_i\Diamond\phi$
Name'	*If* $\vdash i \to \phi$ *then* $\vdash \phi$ *where* i *does not occur in* ϕ
Paste$_\Diamond$	*If* $\vdash @_i\Diamond j \wedge @_j\phi \to \psi$ *and* $j \neq i$ *does not occur in* ϕ *or* ψ *then* $\vdash @_i\Diamond\phi \to \psi$

PROOF. Here are derivations for some of these results.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{@_i\neg(\phi\to\psi)\to @_i\phi}\,(CT, Gen_@, K_@)\qquad \dfrac{}{@_i\neg(\phi\to\psi)\to @_i\neg\psi}\,(CT, Gen_@, K_@)}{@_i\neg(\phi\to\psi)\to @_i\phi\wedge @_i\neg\psi}}{\neg @_i(\phi\to\psi)\to @_i\phi\wedge\neg @_i\psi}\,(Selfdual_@)}{(@_i\phi\to @_i\psi)\to @_i(\phi\to\psi)} \qquad \boxed{K_@^{-1}}$$

$$\dfrac{\dfrac{\dfrac{}{j\to(\phi\leftrightarrow @_j\phi)}\,(Intro)}{@_i j\to(@_i\phi\leftrightarrow @_i @_j\phi)}\,(Gen_@, K_@)\qquad \dfrac{}{@_i @_j\phi\leftrightarrow @_j\phi}\,(Agree)}{@_i j\to(@_i\phi\leftrightarrow @_j\phi)} \qquad \boxed{Nom}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{j\to(i\leftrightarrow @_j i)}\,(Intro)}{@_i j\to(@_i i\leftrightarrow @_i @_j i)}\,(Gen_@, K_@)\qquad \dfrac{}{@_i @_j i\leftrightarrow @_j i}\,(Agree)}{@_i j\to(@_i i\leftrightarrow @_j i)}\qquad \dfrac{}{@_i i}\,(Ref_@)}{@_i j\to @_j i} \qquad \boxed{Sym}$$

$$\dfrac{\dfrac{i\to\phi}{@_i\phi}\,(Gen_@, K_@, Ref_@)}{\phi}\,(Name) \qquad \boxed{Name'}$$

■

With these syntactic preliminaries out of the way, we are ready to start the model construction.

Definition 1. *Let Σ be a set of $\mathcal{H}(@)$ formulas.*

- Σ *is* named *if one of its elements is a nominal.*
- Σ *is* $\Diamond$-saturated *if for all $@_i\Diamond\phi\in\Sigma$, there is a nominal j such that $@_i\Diamond j\in\Sigma$ and $@_j\phi\in\Sigma$.*
- *Let ρ be the existential saturation rule corresponding to the strong Priorean formula $\forall s_1\cdots s_n\exists t_1\cdots t_k.\theta(s_1,\ldots,s_n,t_1,\ldots,t_m)$. Then Σ is ρ-saturated, if for all nominals $i_1\ldots i_n$ there are nominals $j_1\ldots j_m$ such that $\theta(i_1,\ldots,i_m,j_1,\ldots,j_m)\in\Sigma$.*

Lemma 3 (Lindenbaum Lemma). *Every $\mathbf{K}_{\mathcal{H}(@)}+\Lambda+P$ consistent set of formulas can be extended to a named, $\Diamond$-saturated $\mathbf{K}_{\mathcal{H}(@)}+\Lambda+P$ MCS, by adding countably many new nominals to the language.*

Proof. Suppose Σ is $\mathbf{K}_{\mathcal{H}(@)}+\Lambda+P$ consistent. Let $(i_n)_{n\in\mathbb{N}}$ be an enumeration of a countably infinite set of new nominals, and let $(\phi_n)_{n\in\mathbb{N}}$ be an enumeration of the formulas of the extended language. Let Σ^0 denote $\Sigma\cup\{i_0\}$. The *Name′*

rule guarantees that Σ_0 is consistent. For all $n \in \mathbb{N}$, Σ^{n+1} is defined as follows. If $\Sigma^n \cup \{\phi_n\}$ is $\mathbf{K}_{\mathcal{H}(@)} + \Lambda + P$ inconsistent, then $\Sigma^{n+1} = \Sigma^n$. Otherwise:

1. $\Sigma^{n+1} = \Sigma^n \cup \{\phi_n\}$ if ϕ_n is not of the form $@_i\Diamond\psi$.
2. $\Sigma^{n+1} = \Sigma^n \cup \{\phi_n\} \cup \{@_i\Diamond i_m, @_{i_m}\psi\}$ if ϕ_n is of the form $@_i\Diamond\psi$, where i_m is the first new nominal that does not occur in Σ^n or ϕ_n.

Let $\Sigma^\omega = \bigcup_{n\geq 0} \Sigma^n$. Then $\Sigma \subseteq \Sigma^\omega$ and Σ^ω is named, $\Diamond$-saturated, maximal and consistent. The only non-trivial step is in 2, and consistency here is guaranteed by the $Paste_\Diamond$ rule. ∎

LEMMA 4 (Rule Saturation Lemma). *Every $\mathbf{K}_{\mathcal{H}(@)} + \Lambda + P$ consistent set of formulas can be extended to a named, $\Diamond$-saturated, P-saturated $\mathbf{K}_{\mathcal{H}(@)} + \Lambda + P$ MCS, by adding countably many new nominals to the language.*

PROOF. The proof proceeds in two steps. First, we show that every $\mathbf{K}_{\mathcal{H}(@)} + \Lambda + P$ consistent set of formulas Σ can be extended to a set of formulas Σ^+, which is still $\mathbf{K}_{\mathcal{H}(@)} + \Lambda + P$ consistent, such that Σ^+ provides witnesses for Σ, in the following sense: for each existential saturation rule $\rho \in P$ corresponding to a strong Priorean formula $\forall s_1 \cdots s_k \exists t_1 \cdots t_m.\theta$, and for all nominals $i_1, \ldots, i_k$ occurring in Σ, there are nominals $j_1, \ldots, j_k$ such that $\theta(i_1, \ldots, i_k, j_1, \ldots, j_m) \in \Sigma^+$. Such Σ^+ can be constructed as follows.

Let $(i_n)_{n\in\mathbb{N}}$ be an enumeration of a countably infinite set of new nominals, and let $(\rho_n, \vec{i_n})_{n\in\mathbb{N}}$ be an enumeration of all pairs $(\rho_n, i_{n1} \ldots i_{nk})$ where $\rho_n \in P$ is an existential saturation rule for a strong Priorean formula $\forall s_1 \cdots s_k \exists t_1 \cdots t_m.\theta(s_1, \ldots, s_k, t_1, \ldots, t_m)$, and $i_{n1} \ldots i_{nk}$ are nominals occurring in Σ (note that there are at most countably many such pairs). Let $\Sigma^0 = \Sigma$, and for each $n \in \mathbb{N}$, let $\Sigma^{n+1} = \Sigma \cup \{\theta(i_{n1}, \ldots, i_{nk}, j_1, \ldots, j_m)\}$, where ρ_n is the existential saturation rule for the strong Priorean formula $\forall s_1 \cdots s_k \exists t_1 \cdots t_m.\theta$ and $j_1, \ldots, j_m$ are the first m distinct nominals in the enumeration not occurring in Σ^n. Let $\Sigma^+ = \bigcup_n \Sigma^n$. Then $\Sigma \subseteq \Sigma^+$, Σ^+ is $\mathbf{K}_{\mathcal{H}(@)} + \Lambda + P$ consistent and Σ^+ provides witnesses for Σ in the sense described above.

The main argument now runs as follows. Consider any $\mathbf{K}_{\mathcal{H}(@)} + \Lambda + P$ consistent set of formulas Γ. Let $\Gamma^0 = \Gamma$ and for all $n \in \mathbb{N}$, let Γ^{n+1} be a $\Diamond$-saturated named MCS extending $(\Gamma^n)^+$ (the Lindenbaum Lemma guarantees there is one). This gives rise to the following chain of consistent sets of formulas:

$$\Gamma = \Gamma^0 \subseteq (\Gamma^0)^+ \subseteq \Gamma^1 \subseteq (\Gamma^1)^+ \subseteq \cdots$$

Let $\Gamma^\omega = \bigcup_n \Gamma^n$. Then Γ^ω is a $\Diamond$-saturated, named P-saturated MCS. Incidentally, during the entire process we expanded the language with only countably many new nominals, and therefore Γ^ω is a countable set. ∎

Definition 2 (Henkin model obtained from an MCS). *Let* Γ *be a maximal consistent set of* $\mathcal{H}(@)$ *formulas. For all nominals* i, *let* $|i|$ *be* $\{j \mid @_i j \in \Gamma\}$. *Then* $\mathfrak{M}_\Gamma = (W, R, V)$ *is given by*

$$\begin{array}{rcl} W & = & \{|i| \mid i \textit{ is a nominal}\} \\ |i|R|j| & \textit{iff} & @_i\Diamond j \in \Gamma \\ V(p) & = & \{|i| \in W \mid @_i p \in \Gamma\} \\ V(i) & = & \{|i|\} \end{array}$$

That $\mathfrak{M}_\Gamma$ is well-defined follows from *Ref*, *Sym* and *Nom* (note that transitivity is just a special case of *Nom*).

Lemma 5 (Truth Lemma). *For all* $\Diamond$*-saturated MCS's* Γ, *nominals* i *and formulas* ϕ, $\mathfrak{M}_\Gamma, |i| \models \phi$ *iff* $@_i\phi \in \Gamma$

Proof. By induction on the length of ϕ. If ϕ is a proposition letter or nominal, the claim holds by definition. If ϕ is of the form $\psi_1 \to \psi_2$, apply $K_@$ and $K_@^{-1}$. If ϕ is of the form $\neg\psi_1$, apply *Selfdual*$_@$. If ϕ is of the form $@_i\psi$, apply *Agree*. Finally, if ϕ is of the form $\Diamond\psi$, we reason as follows.

Suppose $\mathfrak{M}_\Gamma, |i| \models \Diamond\psi$. Then there is a world $|j|$ such that $|i|R|j|$ and $\mathfrak{M}_\Gamma, |j| \models \psi$. By definition, $@_i\Diamond j \in \Gamma$ and by the induction hypothesis $@_j\psi \in \Gamma$. From this and the derived formula *Bridge*, it follows that $@_i\Diamond\psi \in \Gamma$. Conversely, suppose $@_i\Diamond\psi \in \Gamma$. Then by $\Diamond$-saturation, $@_i\Diamond j \in \Gamma$ and $@_j\psi \in \Gamma$ for some nominal j. By definition, $|i|R|j|$ and by the induction hypothesis, $\mathfrak{M}_\Gamma, |j| \models \psi$. Therefore, $\mathfrak{M}_\Gamma, |i| \models \Diamond\psi$. ■

Lemma 6 (Frame Lemma). *If* Γ *is a* $\Diamond$*-saturated,* P*-saturated* $\mathbf{K}_{\mathcal{H}(@)} + \Lambda + P$ *MCS, then the underlying frame of* $\mathfrak{M}_\Gamma$ *satisfies the frame properties defined by* Λ *and* P.

Proof. Since $\mathfrak{M}_\Gamma$ is a named model and Γ contains all instances of elements of Λ, it follows that the underlying frame of $\mathfrak{M}_\Gamma$ validates Λ. Since $\mathfrak{M}_\Gamma$ is a named model and Γ is P-saturated, it follows that the underlying frame of $\mathfrak{M}_\Gamma$ satisfies (the strong Priorean formulas corresponding to) P. ■

At this point, we have all the required apparatus in place, and we can finish off the proof by the usual kind of argument.

Proof of Theorem 3. Suppose Σ is $\mathbf{K}_{\mathcal{H}(@)} + \Lambda + P$ consistent. By Lemma 4, Σ can be extended to a named, $\Diamond$-saturated, P-saturated MCS Γ. Let $i \in \Sigma$. By Lemma 5, $\mathfrak{M}_\Gamma, |i| \models \Sigma$. By Lemma 6, $\mathfrak{M}_\Gamma$ satisfies all required frame properties. ■

4. A closer look at $\mathbf{K}_{\mathcal{H}(@)}$

Although the basic hybrid language is the weakest language we shall axiomatise, the axiomatic system $\mathbf{K}_{\mathcal{H}(@)}$ we have provided for it and its associated Henkin proof are arguably the most deductively significant. Why? Because the key ideas we require are already present here, and indeed, present in their most novel form. In particular, Theorem 3 and its proof show that hybridisation makes it possible to exploit the classical Henkin completeness technique in a novel way, namely by using nominals in place of first-order constants, and by using rules instead of classical quantifiers, both to provide witnesses and to capture the logic of frame classes beyond the reach of pure axioms. Indeed, as we shall see, although the languages we examine become progressively richer, the axiomatizations required become progressively simpler: we can built straightforwardly on the basis provided by $\mathbf{K}_{\mathcal{H}(@)}$, and indeed simplify some of the underlying machinery.

Given its centrality, it is important to understand $\mathbf{K}_{\mathcal{H}(@)}$ as fully as possible, and in this section we examine it more closely. We first point out two possible variants in the choice of rules. We then show that $\mathbf{K}_{\mathcal{H}(@)}$ enables us to deal with richer modal base languages; in particular, we show hows its ideas can be extended to cover the universal modality and Prior-style tense logic. We then dig deeper. Our use of the Henkin strategy depends on our ability to build named models, and our ability to do this ultimately depends on Name and BG: the linchpin of our approach is Lemma 3, for it is here that we witness $\Diamond$-prefixed formulas. Now, we have already noted that Name and BG are proof-theoretically natural. But are they essential? That is, can we prove analogs Theorems 2 and 3 without their help, or without the help of other non-orthodox rules such as Paste$_\Diamond$? As we shall show, if we want to prove results like Theorems 3 (and indeed, even results like the weaker Theorems 2) for the basic hybrid language, the use of non-orthodox rules is essential.

Alternative choices of axioms and rules

Perhaps surprisingly, the Name and BG rules, together with Gen$_@$, are strong enough to derive the Gen$_\Box$ rule:

$$\dfrac{\dfrac{\dfrac{\dfrac{\phi}{@_j\phi}\ Gen_@}{@_i\Diamond j \to @_j\phi}}{@_i\Box\phi}\ BG}{\Box\phi}\ Name \qquad \boxed{Gen_\Box}$$

This means that in fact, we didn't have to introduce $\mathrm{Gen}_\Box$ separately in the axiomatization in order to obtain completeness. The reason we've included it is, firstly, to emphasise the fact that we are dealing with a normal modal logic, and secondly, to make life slightly easier in the next section, where we will eliminate the BG rule.

A second point worth mentioning is that the Back axiom and the BG rule can be made eliminated at the cost of introducing the $\mathrm{Paste}_\Diamond$ rule.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{@_i @_j \phi \leftrightarrow @_u @_j \phi}\ (Agree, 2x)}{@_u \Diamond i \wedge @_i @_j \phi \to @_u @_j \phi}}{@_u \Diamond @_j \phi \to @_u @_j \psi}\ (Paste_\Diamond)}{\Diamond @_j \phi \to @_j \phi}\ (K_@^{-1}, Name) \qquad \boxed{Back}$$

$$\dfrac{\dfrac{\dfrac{@_i \Diamond j \to @_j \phi}{@_i \Diamond j \wedge @_j \neg\phi \to \bot}\ (Selfdual_@)}{@_i \Diamond \neg\phi \to \bot}\ (Paste_\Diamond)}{@_i \Box \phi}\ (S_@) \qquad \boxed{BG}$$

If we replace Back and BG with the $\mathrm{Paste}_\Diamond$ rule, then the latter can be seen as performing two tasks at the same time: (1) making sure that every accessible world can be consistently named by a nominal, and (2) regulating the interaction between @ and the modalities. Since the elimination of the Name and BG rules will be a theme of this paper, we have decided to keep things simple and put less burden on the rules at this point, which is why we adopt the axiomatization as it is given above. Incidentally, the $\mathrm{Paste}_\Diamond$ rule is also proof-theoretically natural; as is discussed in [2], it is essentially a lightly-disguised sequent rule.

Stronger modal base languages

While we presented the axiomatization of $\mathcal{H}(@)$ only for the case of uni-modal languages, it naturally generalises to multi-modal ones. Of course, in the case of multi-modal languages, the axioms and rules apply to each of the modalities. Given this, extending the completeness proof is a simple exercise.

Axiomatising the universal modality is also straightforward: all we need to do is add the axiom Ei. Completeness follows immediately from the fact that Ei is pure. In a similar fashion, we can axiomatize the *difference modality* (see [16]). The difference modality is a special modality D such that $M, w \models D\phi$ iff there is some world $v \neq w$ such that $M, v \models \phi$. Like the universal modality, this modality can be axiomatized using a pure axiom: $Di \leftrightarrow \neg i$ suffices.

Next, let us consider hybrid tense logic, which uses a basic hybrid language with two diamonds (F and P, whose duals are denoted by G and H respectively), such that F and P are interpreted as each others converse semantically. The simplest way to capture the interaction between F and P is by means of the pure axiom $@_iFj \leftrightarrow @_jPi$. However, if we choose our axioms more carefully, it turns out that the BG rule (both for F and for P) becomes derivable. For this purpose, we need the axioms $@_iGPi$ and $@_iHFi$:

$$\dfrac{\dfrac{}{@_iGPi}\,(TL) \qquad \dfrac{\dfrac{\dfrac{@_iFj \to @_j\phi \qquad \dfrac{\dfrac{}{@_jHFj}\,(TL) \qquad \dfrac{\dfrac{\dfrac{}{HFj \to (Pi \to P(i \wedge Fj))}\,(K, Gen)}{HFj \to (Pi \to @_iFj)}\,(Intro, Back)}{@_jHFj \to (@_jPi \to @_iFj)}}{@_jPi \to @_iFj}}{@_jPi \to @_j\phi}}{Pi \to \phi}\,(K_@^{-1}, Name)}{@_iGPi \to @_iG\phi}\,(G_G, K_G, Gen_@, K_@)}{@_iG\phi} \qquad \boxed{BG_F}$$

The derivation for BG_P is similar. This shows that, in the case of tense languages (that is, languages containing for each modality also the converse modality), one can eliminate BG altogether without losing completeness.

Finally, certain extensions of the language of $\mathcal{H}(@)$ are particularly natural, in that the newly introduced operator can be *locally defined* (that is, definable at named worlds) in terms of the old language. Goranko [21] shows that the Until operator is locally definable in terms of the tense operators: its local definition is $@_i(\phi\mathcal{U}\psi \leftrightarrow F(\psi \wedge H(Pi \to \phi)))$. As one can easily see this gives a complete axiomatization for the Until operator. Another example is the topological closure operator on the reals with $<$, which is locally definable in terms of the tense operators: $@_i(\Diamond\phi \leftrightarrow PG(Fi \to \phi) \vee FH(Pi \to \phi))$. We'll make use of the idea of locally definability when we axiomatise the $\downarrow$ binder.

Non-eliminability of the non-orthodox rules

Theorem 2 covers all pure extensions, but it is based on an axiomatisation that makes use of unorthodox rules (namely Name and BG). Is this necessary, or is the use of such rules avoidable? As we have just noted, we can eliminate the BG if the underlying modal base is tense logic; can an analogous elimination be carried out for a uni-modal modal base language? We shall now show that the answer is *no*. The use of unorthodox rules can only be avoided at the cost of introducing infinitely many rules.

By an orthodox modal rule we mean a rule of the form

$$\frac{\vdash \phi_1(\alpha_1,\ldots,\alpha_n) \quad \& \quad \cdots \quad \& \quad \vdash \phi_k(\alpha_1,\ldots,\alpha_n)}{\vdash \psi(\alpha_1,\ldots,\alpha_n)}$$

Here, $\alpha_1,\ldots,\alpha_n$ range over arbitrary formulas, and are implicitly universally quantified. In the presence of a modus ponens rule (together with enough propositional axioms), we can assume without loss of generality that there is only a single antecedent (a big conjunction), hence all orthodox rules can be assumed to be of the form

$$\frac{\vdash \phi(\alpha_1,\ldots,\alpha_n)}{\vdash \psi(\alpha_1,\ldots,\alpha_n)}$$

The rank of such a rule will be the number of proposition letters occurring in ϕ and ψ (not considering $\alpha_1,\ldots,\alpha_n$), plus n. For example, the rank of the $\mathrm{Gen}_\Box$ rule is 1. A rule *preserves validity* on a class of frames F, if for all formulas $\alpha_1,\ldots,\alpha_n$, $\mathsf{F}\models\phi(\alpha_1,\ldots,\alpha_n)$ implies $\mathsf{F}\models\psi(\alpha_1,\ldots,\alpha_n)$. We can now prove the desired result: no finite collection of orthodox rules can be complete for all pure extensions, even if we take as axioms all validities of $\mathcal{H}(@)$.

THEOREM 4. *Let K_h be the set of all formulas in the basic hybrid language that are valid on all frames, let P be a finite set of orthodox rules, and let L be the axiomatic system formed by taking as axioms K_h, and taking as rules modus ponens, substitution, and all the rules in P. Then there is a pure extension $L+\Lambda$ that is not sound and complete with respect to the class of frames defined by Λ.*

PROOF. Let n be the maximal rank of the rules in P — this information is all we need to construct a pure extension that is incomplete with respect to the frame class it defines. Define Λ be the set containing only the following pure formula:

$$\bigwedge_{1\le l\le 2^n+2} \Diamond i_l \quad\rightarrow \bigvee_{1\le k<l\le 2^n+2} \Diamond(i_k\wedge i_l).$$

$L+\Lambda$ is the axiomatic system L enriched by this single pure axiom (closed under modus ponens, substitution and the rules in P). Let F be the class of frames defined by Λ, that is, the class of all frames in which each world has at most 2^n+1 successors. Either the rules in P preserve validity on F or they do not. If they do not, soundness is lost and there is nothing to prove, so assume that the rules P do preserve validity on F. We shall now show that $L+\Lambda$ is *not* complete for F.

Let M be the class of models that are based on frames in F, and let $\mathcal{F}=(W,R)$ be the frame such that $W=\{1,\ldots,2^n+2\}$ and

$R = W^2$; clearly, $\mathcal{F} \notin \mathsf{F}$. Finally, let M' be $\mathsf{M} \cup \{(\mathcal{F}, V) \mid V$ is a valuation for $\mathcal{F}$ such that $V(i) = V(j)$ for all nominals $i, j\}$. Now for the heart of the proof: we shall show that $L + \Lambda$ is sound for the class of models M'.

CLAIM 1.

- *All formulas in K_h are valid on M'.*
- *Validity on M' is closed under uniform substitution of formulas for propositional variables and nominals for nominals.*
- *Validity on M' is closed under modus ponens*

The proof of Claim 1 is straightforward and is left to the reader.

CLAIM 2. *All formulas valid on F with at most n propositional variables are valid on M'. Hence $\mathsf{M}' \models \Lambda$.*

Let ϕ be a formula with at most n propositional variables, and suppose for the sake of contradiction that $\mathsf{F} \models \phi$ and $\mathsf{M}' \not\models \phi$. Then there is a valuation V and a world w such that $\mathcal{F}, V, w \Vdash \neg\phi$, and such that V assigns the same world to each nominal. Since only n propositional variables occur in ϕ, and all nominals are true at the same world, it follows that the bisimulation contraction of M (over this restricted vocabulary) has at most $2^n + 1$ worlds; hence, this bisimulation contraction is in F. It follows that $\mathsf{F} \not\models \phi$, which contradicts our initial assumption.

CLAIM 3. *All rules in P preserve validity on M'.*

Let $\rho \in P$ be a rule

$$\frac{\vdash \phi(\alpha_1, \ldots, \alpha_m)}{\vdash \psi(\alpha_1, \ldots, \alpha_n)}$$

with $m \leq n$, and suppose that $\mathsf{M}' \models \phi(\alpha_1, \ldots, \alpha_m)$ for particular formulas $\alpha_1, \ldots, \alpha_m$. Uniformly substitute $\top$ for each of the propositional variables occurring in $\alpha_1, \ldots, \alpha_m$. We then obtain pure formulas $\beta_1, \ldots, \beta_m$, and by Claim 1 it follows that $\mathsf{M}' \models \phi(\beta_1, \ldots, \beta_m)$. Let $p_1, \ldots, p_m$ be new, distinct propositional variables. Then it follows that

$$\mathsf{M}' \models \phi((p_1 \lhd \Box\phi(p_1, \ldots, p_m) \rhd \beta_1), \ldots, (p_m \lhd \Box\phi(p_1, \ldots, p_m) \rhd \beta_m))$$

where $(\phi \lhd \psi \rhd \chi)$ is shorthand for $(\psi \wedge \phi) \vee (\neg\psi \wedge \chi)$. Hence

$$\mathsf{F} \models \phi((p_1 \lhd \Box\phi(p_1, \ldots, p_m) \rhd \beta_1), \ldots, (p_m \lhd \Box\phi(p_1, \ldots, p_m) \rhd \beta_m))$$

Since F is a purely definable frame class, ρ preserves validity. Hence, it follows that

$$\mathsf{F} \models \psi((p_1 \lhd \Box\phi(p_1,\ldots,p_m) \rhd \beta_1),\ldots,(p_m \lhd \Box\phi(p_1,\ldots,p_m) \rhd \beta_m))$$

Since this formula contains at most n propositional variables, it follows by Claim 2 that

$$\mathsf{M}' \models \psi((p_1 \lhd \Box\phi(p_1,\ldots,p_m) \rhd \beta_1),\ldots,(p_m \lhd \Box\phi(p_1,\ldots,p_m) \rhd \beta_m))$$

By closure under uniform substitution (Claim 1), it follows that

$$\mathsf{M}' \models \psi((\alpha_1 \lhd \Box\phi(\alpha_1,\ldots,\alpha_m) \rhd \beta_1),\ldots,(\alpha_m \lhd \Box\phi(\alpha_1,\ldots,\alpha_m) \rhd \beta_m))$$

Recall that $\mathsf{M}' \models \phi(\alpha_1,\ldots,\alpha_m)$. It follows that $\mathsf{M}' \models (\alpha_i \lhd \Box\phi(\alpha_1,\ldots,\alpha_m) \rhd \beta_i) \leftrightarrow \alpha_i$. Hence,

$$\mathsf{M}' \models \psi(\alpha_1,\ldots,\alpha_m)$$

This completes the proof of the third claim, and hence we have shown that $L + \Lambda$ is sound with respect to M'

But now the incompleteness result follows. Consider the following formula

$$\eta = \bigwedge_{1\leq i\leq 2^n+2} \Diamond p_i \quad\rightarrow \bigvee_{1\leq i<j\leq 2^n+2} \Diamond(p_i \wedge p_j)$$

Notice that $\mathsf{M}' \not\models \eta$. By Claim 1–3, it follows that $\eta \notin L$. However $\mathsf{F} \models \eta$, so it follows that $L + \Lambda$ is not complete for F. ■

Topological perspective on the non-orthodox rules

The relational semantics for modal logic in terms of Kripke frames is not the only possible semantics. A well known alternative is the *topological semantics* (in fact, historically, it predates Kripke semantics). If we interpret $\Diamond$ as the closure operator, then every class of topological spaces gives rise to a normal modal logic, in fact to an extension of the modal logic **S4**. In [40], the axioms and rules of hybrid logic are considered from the viewpoint of topological semantics. While all axioms of hybrid logic, as well as the Name rule, are still sound under this more general semantics, BG is not. Indeed, even on the most well known topological space, the real line, the BG rule derives invalid conclusions from valid premises. This is in line with the general intuition that the BG rule, with its use of an accessibility assertion, captures the essence of Kripke semantics. Indeed, this intuition can be made precise: as proved in [40], among all topological spaces, BG characterises precisely those that can be represented as a Kripke frame (topologists call such spaces *Alexandroff spaces*).

Incidentally, BG is topologically not only *unsound*, it is also *not needed*. In [40], a general topological completeness result is presented for hybrid logics extending S4 with pure axioms and without the BG rule (but with the universal modality). Similar results can be proved for neighbourhood frames. For a comprehensive treatment of topological *definability* in hybrid logic, see [39].

5. Axiomatizations for $\mathcal{H}(@, \downarrow)$

In this section we turn from $\mathcal{H}(@)$ and its cousins to the richer $\mathcal{H}(@, \downarrow)$. This language, which allows us to bind a world variable to the world of evaluation using the $\downarrow$ binder, has been one of the most extensively explored in contemporary hybrid logic. And quite deservedly, for it turns out to be a key system: it corresponds exactly to the generated submodel invariant fragment of first-order logic [1], and to the intersection of first-order logic and second order propositional modal logic [38]. It is capable of defining any elementary class of frames that is closed under generated subframes and reflects finitely generated subframes [36, 3]. Moreover, having $\downarrow$ in our language, it becomes possible to prove very general interpolation results [1, 8]. Indeed, $\mathcal{H}(@, \downarrow)$ is the smallest possible extension of $\mathcal{H}(@)$ which has interpolation [37].

The language $\mathcal{H}(@, \downarrow)$ was first axiomatized in [12]; here we improve on this earlier work in two ways. First, we will show how to axiomatize $\mathcal{H}(@, \downarrow)$ by adding a single axiom schema to our axiomatization to $\mathcal{H}(@)$. Second, we shall show that it is possible to eliminate the Name and BG rules.

Let's turn to our first axiomatization. We remark that since we're now dealing with a language with variables and binding, we need to adjust the substitution rule to allow variables and nominals to be substituted for each other, and we need to take the standard precautions to prevent accidental binding of variables. Bearing this in mind, consider our first axiomatization, $\mathbf{K}_{\mathcal{H}(@,\downarrow)}$-I, which is shown in Figure 3.

As promised, our first axiomatization extends $\mathbf{K}_{\mathcal{H}(@)}$ with a single axiom schema. Note the form that this schema takes. In the previous section, when we axiomatized the basic hybrid logic of the Until operator, we did so in the way proposed by Goranko: via a local operator definition. The DA schema just given uses essentially the same idea: the schema states the semantics of the $\downarrow$ operator at some arbitrary world named i (the notation $\phi[s := i]$ means substitute the nominal i for all occurrences of the variable s). A more direct axiomatization is hard to imagine, so let's now show that $\mathbf{K}_{\mathcal{H}(@,\downarrow)}$-I really is complete.

$\mathbf{K}_{\mathcal{H}(@,\downarrow)}$-I	
Axioms:	
All axioms of $\mathbf{K}_{\mathcal{H}(@)}$	
DA	$\vdash @_i(\downarrow s.\phi \leftrightarrow \phi[s := i])$
Rules:	
All rules of $\mathbf{K}_{\mathcal{H}(@)}$	

Figure 3. The $\mathbf{K}_{\mathcal{H}(@,\downarrow)}$-I axiomatisation.

We shall state and prove our completeness result for arbitrary sets of sentences. It's easy to adapt the proof to cover arbitrary sets of formulas instead, but the sentential proof is rather elegant.

THEOREM 5 (Completeness). *Let Λ be a set of pure $\mathcal{H}(@,\downarrow)$ axioms and let P be a set of existential saturation rules. A set of $\mathcal{H}(@,\downarrow)$ sentences Σ is $\mathbf{K}_{\mathcal{H}(@,\downarrow)}$-I $+ \Lambda + P$ consistent iff Σ is satisfiable in a model satisfying the frame properties defined by Λ and P.*

PROOF. Almost no changes to our completeness proof for $\mathbf{K}_{\mathcal{H}(@)}$ are required. Once again we build the model out of (equivalence classes of) nominals, and the Lindenbaum Lemma and the Rule Saturation Lemma are unchanged. In fact all we need to do is add an extra clause to our proof of the Truth Lemma for formulas of the form $\downarrow s.\psi$. The extra clause we need is as follows (and this is where we use DA):

$$\begin{array}{lll} \mathfrak{M}_\Gamma, |i| \models \downarrow s.\psi & \Longleftrightarrow & \mathfrak{M}_\Gamma, |i| \models \psi[s := i] \\ & \Longleftrightarrow_{ind.hyp.} & @_i\psi[s := i] \in \Gamma \\ & \Longleftrightarrow_{DA} & @_i\downarrow s.\psi \in \Gamma \end{array}$$

Note that the formulas used in this inductive step are all sentences. Instead of driving the induction directly through the subformula ψ (which may contain the free variable s) we use its sentential variant $\psi[s := i]$. This strategy works because we are evaluating $\downarrow s.\psi$ in a named model and i names the world of evaluation.

With the Truth Lemma established, completeness follows. ■

Before going further, two remarks on existential saturation rules in $\mathcal{H}(@,\downarrow)$. First note that for some frame conditions they are no longer needed. For example, the Church-Rosser property, which could not be defined by any pure $\mathcal{H}(@)$ formula, is defined by the pure $\mathcal{H}(@,\downarrow)$ formula, namely:

$$\Diamond i \wedge \Diamond j \rightarrow @_i(\Diamond\downarrow s.@_j\Diamond s).$$

In fact any elementary class of frames that it definable in ordinary modal logic is definable by pure $\mathcal{H}(@,\downarrow)$-sentences: pure sentences of $\mathcal{H}(@,\downarrow)$ can define all elementary frame classes that are closed under and reflect generated subframes, and all modally definable frame classes have this closure property. However the class of frames in which every world has a predecessor is not definable by means of any pure $\mathcal{H}(@,\downarrow)$ axiom (since it is not closed under generated subframes) so to axiomatize this frame class we would still have to make use of an existential saturation rule.

Secondly, note that in $\mathcal{H}(@,\downarrow)$ we have more such rules at our disposal than in $\mathcal{H}(@)$. Recall that existential saturation rules have the form:

> If $\vdash \phi(i_1,\ldots,i_n,j_1,\ldots,j_m) \to \psi$ then $\vdash \psi$, where $i_1,\ldots,i_n,j_1,\ldots,j_m$ are distinct, and $j_1,\ldots,j_m$ do not occur in ψ.

The only requirement on ψ is that it belong to $\mathcal{H}(@,\downarrow)$, thus we can use rules where ψ contains occurrences of $\downarrow$ and this gives us a wider repertoire, allowing us to define frame properties such as *for each world there is a different world with the same successors*.

Let's turn to our second axiomatisation. While $\mathbf{K}_{\mathcal{H}(@,\downarrow)}$-I is a particularly simple extension of $\mathbf{K}_{\mathcal{H}(@)}$, it inherits from $\mathbf{K}_{\mathcal{H}(@)}$ the use of the Name and BG rules. However, in the richer setting of $\mathcal{H}(@,\downarrow)$ there is no analog of Theorem 4: as we shall now see, we *can* indeed replace the Name and BG rules by axiom schemas (thus answering a question posed in [11]). A second axiomatization, called $\mathbf{K}_{\mathcal{H}(@,\downarrow)}$-II, which works this way, is shown in Figure 4.

$\mathbf{K}_{\mathcal{H}(@,\downarrow)}$-II	
Axioms:	
All axioms of $\mathbf{K}_{\mathcal{H}(@)}$	
DA	$\vdash @_i(\downarrow s.\phi \leftrightarrow \phi[s := i])$
Name$_\downarrow$	$\vdash \downarrow s.(s \to \phi) \to \phi$ provided that s does not occur in ϕ
BG$_\downarrow$	$\vdash @_i\Box\downarrow s.@_i\Diamond s$
Rules:	
MP	If $\vdash \phi$ and $\vdash \phi \to \psi$ then $\vdash \psi$
Subst	If $\vdash \phi$ then $\vdash \phi^\sigma$
Gen$_@$	If $\vdash \phi$ then $\vdash @_i\phi$
Gen$_\downarrow$	If $\vdash \phi$ then $\vdash \downarrow s.\phi$
Gen$_\Box$	If $\vdash \phi$ then $\vdash \Box\phi$

Figure 4. The $\mathbf{K}_{\mathcal{H}(@,\downarrow)}$-II axiomatisation.

Theorem 6 (Completeness). *Let Λ be a set of pure axioms and let P be a set of existential saturation rules. A set of sentences Σ is $\mathbf{K}_{\mathcal{H}(@,\downarrow)}$-II $+ \Lambda + P$ consistent iff Σ is satisfiable in a model satisfying the frame properties defined by Λ and P.*

Proof. By Theorem 5, it suffices to show that the Name and BG rules are $\mathbf{K}_{\mathcal{H}(@,\forall)}$-II derivable.

$$\dfrac{\dfrac{\dfrac{@_i\phi \quad \dfrac{}{i \to (@_i\phi \to \phi)}\,(Intro)}{i \to \phi}}{\downarrow s.(s \to \phi)}\,Subst, G_{\downarrow}}{\phi}\,Name_{\downarrow} \qquad \boxed{Name}$$

$$\dfrac{\dfrac{}{@_i\Box\downarrow s.@_i\Diamond s}\,BG_{\downarrow} \quad \dfrac{\dfrac{\dfrac{@_i\Diamond j \to @_j\phi}{@_j(@_i\Diamond j \to \phi)}\,Gen_@, K_@, S_@}{\dfrac{\downarrow s.@_i\Diamond s \to \phi}{@_i\Box(\downarrow s.@_i\Diamond s \to \phi)}\,Gen_\Box, Gen_@}\,DA, Name}{}}{@_i\Box\phi}\,K_\Box, Gen_@, K_@ \qquad \boxed{BG}$$

Thus $\mathbf{K}_{\mathcal{H}(@,\downarrow)}$-II can derive all axioms and rules of $\mathbf{K}_{\mathcal{H}(@,\downarrow)}$-I, and hence its completeness follows by Theorem 5. ■

A further remark on the eliminability of the Name and BG rules is worth making. They have been eliminated at the expense of introducing two axiom schemes, and one of them, Name$_{\downarrow}$, imposes a non-occurrence requirement. Such restrictions are widely used in axiomatizations of first-order logic, but from an algebraic point of view they are less preferable than normal axiom schemes, or, as algebraic logicians call them, *Monk schemes*. This gives rise to an interesting technical question: can the DA and $Name_{\downarrow}$ schemes be replaced by a finite number of Monk schemes? We conjecture the answer is *no*.

Finally, as we saw in the previous section, if we add the universal modality to the basic hybrid language $\mathcal{H}(@)$, we can axiomatize it simply by adding the single pure formula Ei. Now, it is also possible to add the universal modality to $\mathcal{H}(@,\downarrow)$, which in effect yields the language $\mathcal{H}(A,\downarrow)$, since @ is definable in terms of A. It has long been known (see [9]) that $\mathcal{H}(A,\downarrow)$ is strong enough to define the strong Priorean binders $\exists$ and $\forall$ — so axiomatising this logic is in effect a disguised way of axiomatising $\mathcal{H}(@,\forall)$. The results of this section allow us to do so in two ways: by adding the axiom Ei to either $\mathbf{K}_{\mathcal{H}(@,\downarrow)}$-I or $\mathbf{K}_{\mathcal{H}(@,\downarrow)}$-II. Other axiomatizations for $\mathcal{H}(A,\downarrow)$ are known (notably Goranko's [21] axiomatization, in the paper which introduced $\downarrow$).

$\mathbf{K}_{\mathcal{H}(@,\forall)}$-I	
Axioms:	
All axioms of $\mathbf{K}_{\mathcal{H}(@)}$	
Q1	$\vdash \forall s.(\phi \to \psi) \to (\phi \to \forall s.\psi)$ where s not free in ϕ
Q2	$\vdash \forall s.\phi \to \phi[s := \tau]$ where τ (a world variable or nominal) is substitutable for s in ϕ
$\text{Barcan}_@$	$\vdash \forall s.@_i\phi \leftrightarrow @_i\forall s.\phi$
Rules:	
All rules of $\mathbf{K}_{\mathcal{H}(@)}$	
$\text{Gen}_\forall$	If $\vdash \phi$ then $\vdash \forall s.\phi$

Figure 5. The $\mathbf{K}_{\mathcal{H}(@,\forall)}$-I axiomatisation.

6. Axiomatizations for $\mathcal{H}(@,\forall)$

As we have just remarked, $\mathcal{H}(A,\downarrow)$ is as strong as $\mathcal{H}(@,\forall)$, and we have seen that it is straightforward to provide a complete axiomatization for it. But it is interesting to explore the more direct route to full first-order expressivity provided by the Strong Priorean language $\mathcal{H}(@,\forall)$ and close relatives such as $\mathcal{H}(A,\forall)$. It is true that from a technical perspective, Strong Priorean languages are somewhat less interesting than the weaker hybrid languages we have explored: because they offer quantification over worlds, classical techniques can be applied more straightforwardly. Indeed, as early as 1970, in the first technical paper written on hybrid logic, Bull [14] gave an axiomatisation of $\mathcal{H}(A,\forall)$ that made use of Henkin models. Nonetheless, $\mathcal{H}(@,\forall)$ plays an important role in motivating existential saturation rules, and has considerable historical and philosophical interest (see [6] for a detailed discussion) so let's see how the approach of the present paper deals with it.

We begin by extending our axiomatization for $\mathcal{H}(@)$ to an axiomatization for $\mathcal{H}(@,\forall)$ in the most direct way we know of. We remark that (as with $\mathcal{H}(@,\downarrow)$) we need to adjust the substitution rule so that variables can be substituted for nominals and vice versa, and that we need to observe all the standard precautions to avoid performing illegal substitutions. This done, we define the axiom system shown in Figure 5, which we call $\mathbf{K}_{\mathcal{H}(@,\forall)}$-I.

Conceptually, $\mathbf{K}_{\mathcal{H}(@,\forall)}$-I is very simple system. In essence we have bolted a classic axiomatization for first-order logic (namely Q1, Q2 and $\text{Gen}_\forall$), together with a hybrid analog of the famous Barcan formula (and its converse) familiar from first-order modal logic, onto our axiomatization of the basic hybrid language. Both of these components have a clearly defined role to play when it comes to proving completeness, as we shall now see.

THEOREM 7 (Completeness). *A set of sentences is* $\mathbf{K}_{\mathcal{H}(@,\forall)}$-I *consistent iff it is satisfiable.*

PROOF. The proof involves only modest adjustments to our completeness proof for $\mathbf{K}_{\mathcal{H}(@)}$; indeed in certain respects it is simpler. Once again we build the model out of (equivalence classes of) nominals. The key change is to reformulate the Lindenbaum Lemma (Lemma 3) so that new nominals are introduced to witness the existential quantifiers over worlds. That is, we add the following clause to our inductive Lindenbaum definition:

$\Sigma^{n+1} = \Sigma^n \cup \{\phi_n\} \cup \{\psi[s := i_m]\}$ if ϕ_n is of the form $\exists s.\psi$,
where i_m is the first new nominal that does not occur in Σ^n or ϕ_n.

The consistency of this step is guaranteed by Q1 and $\text{Gen}_\forall$, for with their help it is easy to show that if $\vdash \phi[s/i] \rightarrow \psi$ then $\vdash \exists s.\phi \rightarrow \psi$ (where i does not occur in ϕ or ψ). The analogy between this part of the construction, and the way first-order constants are used in Henkin completeness proofs for first-order logic to witness existential quantifiers, is clear.

The Barcan (and converse Barcan) analog comes into play when proving the Truth Lemma. As in Lemma 5 we want to show that for all nominals i and sentences ϕ, $\mathfrak{M}_\Gamma, |i|_\Gamma \models \phi$ iff $@_i\phi \in \Gamma$. Again we do this by induction, and the only new case is for formulas ϕ of the form $\forall s.\psi$. Now — with one exception — the following equivalences are easy to establish:

1. $@_i \forall s.\phi \in \Gamma$
2. $\forall s.@_i\phi \in \Gamma$
3. $@_i\phi[s := j] \in \Gamma$ for all nominals j
4. $\mathfrak{M}_\Gamma, |i| \models \phi[s := j]$ for all nominals j
5. $\mathfrak{M}_\Gamma, |i| \models \forall s.\phi$

The exception is the equivalence of (1) and (2). This is where we need the Barcan analog, which legitimises the required permutations of @ and $\forall$.

With the Lindenbaum Lemma and the Truth Lemma established, the completeness proof goes through in the expected way. ∎

Note that this basic completeness result is all we need: there is no need to talk about pure extensions, or existential saturation rules. Because we have full first-order expressive power at our disposal, any frame class that can be

$\mathbf{K}_{\mathcal{H}(@,\forall)}$-II	
Axioms:	
All axioms of $\mathbf{K}_{\mathcal{H}(@)}$	
Q1	$\vdash \forall s.(\phi \to \psi) \to (\phi \to \forall s.\psi)$ where s not free in ϕ
Q2	$\vdash \forall s.\phi \to \phi[s := i]$ where i is substitutable for s in ϕ
$\text{Barcan}_{@}$	$\vdash \forall s.@_i\phi \leftrightarrow @_i\forall s.\phi$
$\text{Ref}_{\exists}$	$\vdash \exists s.s$
$\text{Barcan}_{\Box}$	$\vdash \forall s\Box\phi \leftrightarrow \Box\forall s\phi$
Rules:	
MP	If $\vdash \phi$ and $\vdash \phi \to \psi$ then $\vdash \psi$
Subst	If $\vdash \phi$ then $\vdash \phi^{\sigma}$
$\text{Gen}_{@}$	If $\vdash \phi$ then $\vdash @_i\phi$
$\text{Gen}_{\forall}$	If $\vdash \phi$ then $\vdash \forall s.\phi$
$\text{Gen}_{\Box}$	If $\vdash \phi$ then $\vdash \Box\phi$

Figure 6. The $\mathbf{K}_{\mathcal{H}(@,\forall)}$-II axiomatisation.

axiomatized using pure axioms or existential saturation rules (indeed, any first-order definable frame-class whatsoever) can be described by a $\mathcal{H}(@,\forall)$ sentence (this was first observed by Bull [14] in his pioneering paper). So extended completeness for all elementary frame classes is an immediate corollary of the basic completeness result just given.

The $\mathbf{K}_{\mathcal{H}(@,\forall)}$-I axiomatization is direct, and makes the link with our work on $\mathcal{H}(@)$ clear. Nonetheless it makes use of the Name and BG rules, and with the expressive power at our disposal it is possible to eliminate these rules in an elegant way: we add the following axiom $\exists s.s$ (which says, in essence, that it is always possible to name the current world) and the schema $\forall s\Box\phi \leftrightarrow \Box\forall s\phi$, another Barcan and converse Barcan analog, this time one which permits $\forall$ and $\Box$ to be permuted. We explicitly list the axioms and rules of the second axiomatization (which we shall call $\mathbf{K}_{\mathcal{H}(@,\forall)}$-II) in Figure 6.

Theorem 8 (Completeness). *A set of formulas is* $\mathbf{K}_{\mathcal{H}(@,\forall)}$-II *consistent iff it is satisfiable.*

Proof. We prove this by showing that the Name and BG rules are $\mathbf{K}_{\mathcal{H}(@,\forall)}$-II derivable. As a first step, we show that all instances of a formula called $\text{Name}_{\forall}$ are $\mathbf{K}_{\mathcal{H}(@,\forall)}$-II derivable:

$$\dfrac{\dfrac{}{\exists s.s}\,(Ref_\exists) \qquad \dfrac{\dfrac{}{\forall s.(\neg\phi \to \neg s) \to (\neg\phi \to \forall s.\neg s)}\,(Q1)}{\forall s.(s \to \phi) \to (\exists s.s \to \phi)}}{\forall s.(s \to \phi) \to \phi \quad (\text{where } s \text{ does not occur freely in } \phi)} \qquad \boxed{Name_\forall}$$

Now we can show that the Name rule is $\mathbf{K}_{\mathcal{H}(@,\forall)}$-II derivable:

$$\dfrac{\dfrac{\dfrac{\dfrac{@_i\phi}{i \to \phi}\,(Intro)}{s \to \phi}\,(Subst)}{\forall s.(s \to \phi)}\,(G_\forall) \qquad \dfrac{}{\forall s.(s \to \phi) \to \phi}\,(Name_\forall)}{\phi} \qquad \boxed{Name}$$

Finally, we can show that the BG rule is $\mathbf{K}_{\mathcal{H}(@,\forall)}$-II derivable. The proof makes the fundamental role of the two Barcan analogs clear:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{@_i\Diamond j \to @_j\phi}{@_i\Diamond j \to j \to \phi}\,(Intro)}{@_i\Diamond j \to \Box j \to \phi}\,(Back, S_@)}{@_i(\Diamond j \to @_i\Box(j \to \phi))}\,(Nec_@, K_@, Agree)}{@_i(\Diamond j \to \Box(j \to \phi))}\,(K_@^{-1})}{@_i\Box(j \to \phi)}\,(K_\Box, K_@, Gen_@)}{@_i\Box(s \to \phi)}\,(Subst)}{\forall s.@_i\Box(s \to \phi)}\,(G_\forall)}{@_i\forall s.\Box(s \to \phi}\,(Barcan_@)}{@_i\Box\forall s.(s \to \phi)}\,(Barcan_\Box) \qquad \dfrac{\dfrac{\dfrac{}{\forall s.(s \to \phi) \to \phi}\,(Name_\forall)}{\Box\forall s.(s \to \phi) \to \Box\phi}\,(Gen_\Box, K_\Box)}{@_i\Box\forall s.(s \to \phi) \to @_i\Box\phi}\,(Gen_@, K_@)}{@_i\Box\phi} \qquad \boxed{\text{BG}}$$

Thus $\mathbf{K}_{\mathcal{H}(@,\forall)}$-II can prove everything that $\mathbf{K}_{\mathcal{H}(@,\forall)}$-I can, hence its completeness follows by Theorem 7. ■

As far as we are aware, $\mathbf{K}_{\mathcal{H}(@,\forall)}$-I and $\mathbf{K}_{\mathcal{H}(@,\forall)}$-II are the first axiom systems to be given for the language $\mathcal{H}(@,\forall)$. However $\mathcal{H}(@,\forall)$ is very similar to the language $\mathcal{H}(A,\forall)$, and as we have already mentioned, in a 1970 paper Bull provided a complete axiomatization for this language (actually, Bull's work is in the somewhat richer setting of hybrid tense logic, but this makes little difference to the proof details). Bull's axiomatization is similar to $\mathbf{K}_{\mathcal{H}(@,\forall)}$-II. In particular, Bull makes use of the axiom $\exists s.s$ and the Barcan analog $\forall s.A\phi \leftrightarrow A\forall s.\phi$ (which permits the universal modality and the $\forall$ binder to be permuted). Moreover, Bull's completeness proof is also a Henkin-style completeness proof.

7. An axiomatization of $\mathcal{QHL}(@)$

Till now we have worked solely with propositional hybrid logics. But the methods we have discussed are robust, and to demonstrate this we now apply them to the hybrid analogue of quantified modal logic. Recent work (see [2]) has shown that quantified hybrid logic is well behaved with respect to the interpolation property (far better behaved than orthodox quantified modal logic). Our goal here is to show that the same holds true of its axiomatics. As we shall see, the Henkin-style model construction used for $\mathcal{H}(@)$ can be straightforwardly adapted to the richer setting. Moreover, the general completeness results for extensions with pure axioms and existential saturation rules become even more useful, since they can be used not only to express frame properties but also a wide range of domain conditions.

The language of quantified hybrid logic we shall work with is called $\mathcal{QHL}(@)$ and it is built on top of $\mathcal{H}(@)$. Let us assume that we have fixed a set of constant symbols $(c, d, \ldots)$ and a set of relation symbols (R, P, $\ldots$). In what follows we assume that these sets are at most countably infinite and that the arity of each relation symbol is known. We keep all the ordinary proposition letters from $\mathcal{H}(@)$, but from now on we regard these as zero-ary relation symbols. In addition to these non-logical symbols we shall need a countably infinite set VAR of first-order variables $(x, y, z, \ldots)$, the equality symbol $=$, and the quantifier symbol $\forall$. Please note: in this section $\forall$ will be used in the traditional first-order way (that is, as a quantifier over ordinary individuals) not as a Prior-style hybrid quantifier over worlds!

There are three kinds of terms in $\mathcal{QHL}(@)$: first-order variables (x, y, z and so on), constants (c, d, e and so on), and *rigidified constants* ($@_i c, @_j d$, and so on). Intuitively, $@_i c$ rigidly refers to the denotation of the non-rigid constant c in the world named by i, and the semantics given below will make this precise. Note that we are (deliberately) overloading the @ symbol. Till now @ was used as an operator on formulas, whereas now we are using it as an operator on first-order constants. From the perspective of hybrid logic this is a natural notational choice: the idea of hybrid logic rests on the idea of using propositions as terms, and our overloading of @ is much in the same spirit. Readers who dislike this should introduce a different rigidification symbol (say δ) and adjust the axiomatic system given below accordingly. In what follows we write σ, τ, κ, and so on, for terms (variables, constants or rigidified constants). A term is called *rigid* if it is either a variable of a rigidified constant.

The formulas of $\mathcal{QHL}$ are defined as follows

$$\phi ::= R\vec{\tau} \mid \tau = \kappa \mid i \mid \neg\phi \mid \phi \wedge \psi \mid \Diamond\phi \mid \forall x.\phi \mid @_i\phi$$

In what follows, we make free use the usual defined symbols (such as $\exists$, $\Box$ and $\rightarrow$). In addition we use $\textsc{exists}(\tau)$ as a shorthand for $\exists y.y = \tau$, where y is distinct from τ. As usual, a sentence is a formula without free variables.

Now for the semantics. Skeletons are the first-order generalisation of frames: a skeleton provides the interpretation of the modalities and furthermore, it assigns to each world a domain of entities.

DEFINITION 3 (Skeleton). *A skeleton is a structure* $\mathfrak{S} = (W, Dom, D, R)$*, where* W *is a set of worlds,* Dom *is a set of entities,* $D : W \rightarrow \wp(Dom)$ *assigns to every world a non-empty subset of* Dom *and* $R \subseteq W^2$*.*

DEFINITION 4 (Models). *A model is a structure* $\mathfrak{M} = (\mathfrak{S}, I)$*, where* $\mathfrak{S} = (W, Dom, D, R)$ *is a skeleton and* I *interprets the nominals, constants and predicates. More precisely,* $I(i) \in W$ *for all nominals* i*,* $I_w(c) \in Dom$ *for all constants* c*, and* $I_w(P) \subseteq Dom^n$ *for all predicates* P *of arity* n*. Note that* I *supplies the interpretation of the propositional letters* p*,* q*,* r*, and so on, as these are now regarded as zero-ary predicates.*

DEFINITION 5 (Semantics). *Let* $\mathfrak{M} = (W, Dom, D, R, I)$ *be any model. A function* $g : VAR \longrightarrow Dom$ *is called an assignment on* $\mathfrak{M}$*. Given a model* $\mathfrak{M}$ *and an assignment* g *on* $\mathfrak{M}$ *we interpret the terms of* $\mathcal{QHL}(@)$ *as follows:*

$$[\tau]^{\mathfrak{M},w,g} = \begin{cases} g(x) & \text{if } \tau \text{ is the variable } x \\ I_w(c) & \text{if } \tau \text{ is of the form } c \\ I_v(c) & \text{if } \tau \text{ is of the form } @_i c \text{ and } I(i) = v \end{cases}$$

The satisfaction definition for arbitrary formulas is then defined as follows:

$$\begin{array}{lll}
\mathfrak{M}, w, g \models \tau_1 = \tau_2 & \textit{iff} & [\tau_1]^{\mathfrak{M},w,g} = [\tau_2]^{\mathfrak{M},w,g} \\
\mathfrak{M}, w, g \models P\tau_1 \dots \tau_n & \textit{iff} & ([\tau_1]^{\mathfrak{M},w,g}, \dots, [\tau_n]^{\mathfrak{M},w,g}) \in I_w(P) \\
\mathfrak{M}, w, g \models i & \textit{iff} & I(i) = w \\
\mathfrak{M}, w, g \models \neg\phi & \textit{iff} & \mathfrak{M}, w, g \not\models \phi \\
\mathfrak{M}, w, g \models \phi \wedge \psi & \textit{iff} & \mathfrak{M}, w, g \models \phi \textit{ and } \mathfrak{M}, w, g \models \psi \\
\mathfrak{M}, w, g \models \Diamond\phi & \textit{iff} & \textit{there is a } v \in W \textit{ such that } wRv \textit{ and } \mathfrak{M}, v, g \models \phi \\
\mathfrak{M}, w, g \models \forall x.\phi & \textit{iff} & \mathfrak{M}, w, g[x \mapsto d] \models \phi \textit{ for all } d \in D_w \\
\mathfrak{M}, w, g \models @_i\phi & \textit{iff} & \mathfrak{M}, v, g \models \phi \textit{ where } I(i) = \{v\}
\end{array}$$

Note that the interpretation of constants and predicates in a world w is not restricted to the local domain D_w. Thus, even non-existent entities can have various properties and stand in various relationships with each other. The quantifiers, on the other hand, receive an *actualist* interpretation: they quantify over elements of the local domain. Equality is rigid: it is the same relation in every world. Just as we defined validity of $\mathcal{H}(@)$ formulas on frames, we say that a formula ϕ of $\mathcal{QHL}(@)$ is valid on a skeleton $\mathfrak{S}$ if $(\mathfrak{S}, I), w, g \models \phi$ for all choices of I, w, and g.

Now for the axiomatization. First of all, recall the substitution rule in the axiomatization of $\mathcal{H}(@)$, which allows us to replace proposition letters by arbitrary formulas and nominals by nominals. In the case of $\mathcal{QHL}(@)$, the proposition letters are now regarded as zero-ary predicates. Therefore, the substitution rule now allows us to uniformly replace zero-ary predicates by arbitrary formulas and nominals by nominals. Furthermore, the rule is extended in such a way that rigidified constants can be uniformly substituted for variables and vice versa. Having made these changes to the substitution rule, the axiomatization for $\mathcal{QHL}(@)$ is as shown in Figure 7.

$\mathbf{K}_{\mathcal{QHL}(@)}$	
Axioms:	
All axioms of $\mathbf{K}_{\mathcal{H}(@)}$	
Q1	$\vdash \forall x.(\phi \to \psi) \to (\phi \to \forall x.\psi)$ where x not in ϕ
Q2	$\vdash \forall x.\phi \to (\text{EXISTS}(\tau) \to \phi[x := \tau])$ where τ is rigid.
Q3	$\vdash \forall x.\text{EXISTS}(x)$
Ref$_=$	$\vdash \tau = \tau$
Repl$_=$	$\vdash \sigma = \tau \to (\phi(\sigma) \to \phi(\tau))$ where σ, τ are rigid
Rigidify	$\vdash @_i(c = @_i c)$
NED	$\vdash \exists x.\top$
Rules:	
All rules of $\mathbf{K}_{\mathcal{H}(@)}$	
Gen$_\forall$	If $\vdash \phi$ then $\vdash \forall x.\phi$

Figure 7. The $\mathbf{K}_{\mathcal{QHL}(@)}$ axiomatisation.

The completeness result for $\mathbf{K}_{\mathcal{QHL}(@)}$ that we will prove is a natural generalisation of Theorem 3 for $\mathbf{K}_{\mathcal{H}(@)}$. In the setting of quantified hybrid logic, we call a formula pure if it contains no relation symbols (of any arity, not just the zero-ary relation symbols) and no non-rigidified constants. That is, rigidified constants, variables, nominals and the equality symbol are allowed in pure formulas. As in the propositional case, existential saturation rules have the form:

> If $\vdash \phi(i_1, \ldots, i_n, j_1, \ldots, j_m) \to \psi$ then $\vdash \psi$, where $i_1, \ldots, i_n, j_1, \ldots, j_m$ are distinct, and $j_1, \ldots, j_m$ do not occur in ψ.

where ψ is a pure formula.

Incidentally, the prohibition on non-rigidified constants appearing in pure formulas (which also prevents them from appearing in existential saturation rules) is crucial to this theorem. Since constants are interpreted semantically as functions from worlds to individuals, axioms containing non-rigidified constants are essentially second-order in nature: they express universal quantification over functions. Therefore, we cannot expect general completeness results for axioms involving such constants. Hence our insistence that pure formulas only contain rigidified constants.

THEOREM 9 (Completeness). *If Σ is a $\mathbf{K}_{\mathcal{QHL}(@)} + \Lambda + P$ consistent set of sentences, where Λ is a set of pure axioms and P is a set of existential saturation rules, then Σ is satisfiable in a model satisfying the skeleton properties defined by Λ and P.*

Before we proceed with the proof of Theorem 9, note that it can be applied to domain conditions, and not merely frame conditions. For example, pure axioms can be used to characterise a number of standard (and not so standard) domain conditions:

Increasing domains:	$\textsc{exists}(@_k c) \to \Box\textsc{exists}(@_k c)$
Decreasing domains:	$\Diamond\textsc{exists}(@_k c) \to \textsc{exists}(@_k c)$
Constant domains:	$@_i\textsc{exists}(@_k c) \to @_j\textsc{exists}(@_k c)$
Full domains:	$\textsc{exists}(@_k c)$
Disjoint domains:	$@_i\textsc{exists}(@_k c) \land @_j\textsc{exists}(@_k c) \to @_i j$
Convex domains:	$\textsc{exists}(@_k c) \to \Box(\Diamond\textsc{exists}(@_k c) \to \textsc{exists}(@_k c))$

To give a more elaborate example, suppose our modality models the flow of time, and our individuals are humans. Then it would be natural to consider structures in which the accessibility relation is a strict total order that extends infinitely to the past and future, and for every individual e there are two time-points w and v (birth and death) such that e exists in all and only the worlds in between w and v. Notice that all these requirements can be naturally formulated using pure axioms (provided that we work with a language containing the Priorean past tense operator P).

Let us turn to the completeness proof. First some syntactic preliminaries:

LEMMA 7. *The following schemas are all derivable in $\mathbf{K}_{\mathcal{QHL}(@)}$:*

$Sym_=$	$\vdash \sigma = \tau \to \tau = \sigma$
$Trans_=$	$\vdash \sigma = \kappa \to \kappa = \tau \to \sigma = \tau$
Nom_c	$\vdash @_i j \to (@_i c = @_j c)$
$Agree_=$	$\vdash @_i(\sigma = \tau) \leftrightarrow \sigma = \tau$, *where* σ, τ *are variables or rigidified constants*
$Paste_\forall$	*If* $\vdash @_i\textsc{exists}(c) \wedge @_i\phi[x := @_i c] \to \psi$ *then* $\vdash @_i \exists x.\phi \to \psi$ *where* c *does not occur in* ψ

PROOF.

$$\dfrac{\dfrac{}{c = d \to c = c \to d = c}\,(Eq, Subst) \qquad \dfrac{}{c = c}\,(Ref_=)}{c = d \to d = c} \qquad \boxed{Sym_=}$$

$$\dfrac{\dfrac{}{d = e \to c = d \to c = e}\,(Eq, Subst)}{c = d \to d = e \to c = e} \qquad \boxed{Trans_=}$$

$$\dfrac{\dfrac{}{@_i c = @_i c}\,(Ref_=) \qquad \dfrac{}{@_i j \to (@_i c = @_i c) \to (@_i c = @_j c)}\,(Nom, Subst)}{@_i j \to (@_i c = @_j c)} \qquad \boxed{Nom_c}$$

$$\dfrac{\dfrac{\dfrac{}{@_i(\tau = \tau)}\,Ref_=, Gen_@}{\tau = \kappa \to @_i(\tau = \kappa)}\,Repl = \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{@_j(\tau = \tau)}\,Ref_=, Gen_@}{\tau = \kappa \to @_j(\tau = \kappa)}\,Repl_=}{@_i(\tau = \kappa \to @_j(\tau = \kappa))}\,Gen_@}{@_i(\tau = \kappa) \to @_j(\tau = \kappa}\,K_@, Agree}{@_j(@_i(\tau = \kappa) \to \tau = \kappa)}\,K_@^{-1}}{@_i(\tau = \kappa) \to \tau = \kappa}\,Name}{@_i(\tau = \kappa) \leftrightarrow \tau = \kappa} \qquad \boxed{Agree_=}$$

$$\begin{array}{cl}
@_i\textsc{exists}(c) \wedge @_i\phi[x := @_i c] \to \psi & \\ \hline
@_i\textsc{exists}(c) \wedge @_i\phi[x := @_i c] \to @_j\psi & (Gen_@, K_@, Agree) \\ \hline
i \wedge \textsc{exists}(c) \wedge \phi \to @_j\psi & (Intro) \\ \hline
\forall x.(i \wedge \textsc{exists}(x) \wedge \phi \to @_j\psi) & (G_\forall, Subst) \\ \hline
i \wedge \exists x.(\textsc{exists}(x) \wedge \phi) \to @_j\psi & (Q1) \\ \hline
i \wedge \exists x.\phi \to @_j\psi & (Q3, Q1) \\ \hline
@_i\exists x.\phi \to @_j\psi & (Intro, Ref_@, Agree) \\ \hline
@_j(@_i\exists x.\phi \to \psi) & (K_@^{-1}, Agree) \\ \hline
@_i\exists x.\phi \to \psi & (Name)
\end{array} \qquad \boxed{Paste_\forall}$$

■

We are now ready for the model construction. As promised, all we have to do is adapt the Henkin construction used in the completeness proof for $\mathcal{H}(@)$.

Definition 6 ($\exists$-Saturation). *A set of sentences* Σ *is* $\exists$-saturated *if whenever* $@_i\exists x.\phi \in \Sigma$*, there is a constant* c *such that we have both* $@_i\textsc{exists}(c) \in \Sigma$ *and* $@_i\phi[x := @_ic] \in \Sigma$.

Lemma 8 (Extended Lindenbaum Lemma). *Every* $\mathbf{K}_{qhl(@)}+\Lambda+P$ *consistent set of sentences can be extended to a named,* $\Diamond$*-saturated,* $\exists$*-saturated,* $\mathbf{K}_{\mathcal{QHL}(@)} + \Lambda + P$ *MCS.*

Proof. Suppose Σ is consistent. Let $(i_n)_{n\in\mathbb{N}}$ be an enumeration of a countably infinite set of new nominals, let $(c_n)_{n\in\mathbb{N}}$ be an enumeration of a countably infinite set of new nominals, and let $(\phi_n)_{n\in\mathbb{N}}$ be an enumeration of the formulas of the extended language. Let Σ^0 denote $\Sigma \cup \{i_0\}$. The *Name'* rule guarantees that Σ_0 is consistent. For all $n \in \mathbb{N}$, Σ^{n+1} is defined as follows. If $\Sigma^n \cup \{\phi_n\}$ is inconsistent, then $\Sigma^{n+1} = \Sigma^n$. Otherwise:

1. $\Sigma^{n+1} = \Sigma^n \cup \{\phi_n\}$ if ϕ_n is not of the form $@_i\Diamond\psi$ or $\exists x.\psi$.

2. $\Sigma^{n+1} = \Sigma^n \cup \{\phi_n\} \cup \{@_i\Diamond i_m, @_{i_m}\psi\}$ if ϕ_n is of the form $@_i\Diamond\psi$,
   where i_m is the first new nominal that does not occur in Σ^n or ϕ_n.

3. $\Sigma^{n+1} = \Sigma^n \cup \{\phi_n\} \cup \{@_i\textsc{exists}(c_m), @_i\psi[x := @_ic_m]\}$ if ϕ_n is of the form $@_i\exists x\psi$,
   where c_m is the first new constant that does not occur in Σ^n or ϕ_n.

Let $\Sigma^+ = \bigcup_{n\geq 0}\Sigma^n$. Then $\Sigma \subseteq \Sigma^+$ and Σ^+ is named, $\Diamond$-saturated, $\exists$-saturated, maximal and consistent. The only non-trivial steps are in 2. and 3., and the consistency of these step is guaranteed by the $Paste_\Diamond$ rule and the $Paste_\forall$ rule, respectively. ∎

Lemma 9 (Extended Rule Saturation Lemma). *Every* $\mathbf{K}_{\mathcal{QHL}(@)} + \Lambda + P$ *consistent set of sentences can be extended to a named,* $\Diamond$*-saturated,* $\exists$*-saturated,* P*-saturated* $\mathbf{K}_{\mathcal{QHL}(@)} + \Lambda + P$ *MCS.*

Proof. As in the proof of Lemma 4. ∎

Definition 7 (Henkin model obtained from an MCS). *Let* Γ *be a* $\mathbf{K}_{\mathcal{QHL}(@)}$ *MCS. For all nominals* i*, let* $|i| = \{j \mid @_ij \in \Gamma\}$*. For all rigidified constants* τ*, let* $|\tau| = \{\kappa \mid \kappa$ *is a rigidified constant and* $(\tau = \kappa) \in \Gamma\}$*. The model*

$\mathfrak{M}_\Gamma = (W, Dom, D, R, I)$ *is defined as follows.*

$$\begin{array}{rcl} W & = & \{|i| \mid i \textit{ is a nominal}\} \\ Dom & = & \{|\tau| \mid \tau \textit{ is a rigidified constant}\} \\ D_{|i|} & = & \{|\tau| \in Dom \mid @_i\text{EXISTS}(\tau) \in \Gamma\} \\ |i|R|j| & \textit{iff} & @_i\Diamond j \in \Gamma \\ I(i) & = & \{|i|\} \\ I_{|i|}(c)(|i|) & = & |@_i c| \\ I_{|i|}(R) & = & \{(|\tau_1|, \ldots, |\tau_n|) \in D_{|i|}^{arity(R)} \mid @_i R\tau_1 \ldots \tau_n \in \Gamma\} \end{array}$$

That R is well-defined follows from *Ref*, *Sym* and *Nom* (as before). That D is well defined follows from $Ref_=$, $Sym_=$ and $Trans_=$ together with $Repl_=$. That the extension assigned to constants by I is well-defined follows from Nom_c. That the extension assigned to predicates by I is well-defined is guaranteed by *Nom* and $Repl_=$.

LEMMA 10 (Truth Lemma). *For all* $\Diamond$*-saturated,* $\exists$*-saturated MCS's* Γ*, nominals* i *and sentences* ϕ*,* $\mathfrak{M}_\Gamma, |i| \models \phi$ *iff* $@_i\phi \in \Gamma$

PROOF. As before, by induction on the length of ϕ. We will only discuss the inductive step for atomic formulas and for the formulas of the form $\forall x.\psi$.

If ϕ is an atomic formula, then it must be of the form $\tau = \kappa$, $R\vec{\tau}$ or i. In either case, the claim follows immediately from the definition of $\mathfrak{M}_\Gamma$ (via the Rigidify axiom in case of non-rigidified constants).

If ϕ is of the form $\forall x.\psi$, we reason as follows. Suppose $\mathfrak{M}_\Gamma, |i| \models \forall x.\psi$. Then by a simple inductive argument, one can show that for all rigidified constants τ, if $|\tau| \in D_{|i|}$ then $\mathfrak{M}_\Gamma, |i| \models \psi[x := \tau]$. By our induction hypothesis, it follows that for all rigidified constants τ, if $@_i\text{EXISTS}(\tau) \in \Gamma$ then $@_i\psi[x := \tau] \in \Gamma$. Since Γ is $\exists$-saturated, it follows that $@_i\exists x.\neg\psi \notin \Gamma$ and therefore $@_i\forall x.\psi \in \Gamma$. Conversely, suppose $@_i\forall x.\psi \in \Gamma$. Then for all rigidified constants τ, if $@_i\text{EXISTS}(\tau) \in \Gamma$ then $@_i\psi[x := \tau] \in \Gamma$ (by axiom Q2). Since every element of $D_{|i|}$ is named by a constant, it follows (using the induction hypothesis) that $\mathfrak{M}_\Gamma, |i| \models \forall x.\psi$. ■

Finally, all the apparatus is in place, and we can finish off the main completeness argument as follows.

PROOF OF THEOREM 9. Let Σ be any $\mathbf{K}_{\mathcal{QHL}(@)} + \Lambda + P$ consistent set of sentences. By Lemma 9, Σ can be extended to a named, $\Diamond$-saturated, $\exists$-saturated, P-saturated MCS Γ, and let $i \in \Gamma$. By the Lemma 10, $\mathfrak{M}_\Gamma, |i| \models \Sigma$.

Since $\mathfrak{M}_\Gamma$ is a named model and Γ contains all instances of elements of Λ, it follows that the underlying frame of $\mathfrak{M}_\Gamma$ validates Λ. Since $\mathfrak{M}_\Gamma$ is a named model and Γ is P-saturated, it follows that the underlying frame of $\mathfrak{M}_\Gamma$ admits P. ■

To conclude, two remarks. We have here axiomatized the basic quantified hybrid logic $\mathcal{QHL}(@)$, that is, the quantified hybrid logic built over $\mathcal{H}(@)$. However we could have built quantified hybrid logic over $\mathcal{H}(@, \downarrow)$ or even $\mathcal{H}(@, \forall)$, thus obtaining $\mathcal{QHL}(@, \downarrow)$ and $\mathcal{QHL}(@, \forall)$ respectively. Although we won't prove this, it will be clear to readers who have worked through the details of the completeness proofs that we can combine the completeness result just given with the completeness results for $\mathcal{H}(@, \downarrow)$ or even $\mathcal{H}(@, \forall)$.

There are many ways to define a semantics for quantified modal/hybrid logic, and there is no consensus concerning the nicest semantics. It is worth mentioning that both our results and our methods are fairly robust. For example, it would have been just as easy to start with a system of constant domains and add a separate existence predicate. There is however one particular interesting point. Melvin Fitting has defined a varying domain semantics for quantified modal logic in which it is required that every object is in the domain of some world. Unlike the domain conditions discussed so far, this particular condition cannot be captured by means of a pure axiom. It is not hard to see that the basic quantified modal logic of the class of all frames is complete with respect to this alternative semantics as well. However, for axiomatic extensions this need no longer be the case. Existential saturation rules provide the solution to this problem: the class of frames in which every object is in the domain of some world is characterised by the existential saturation rule *If* $\vdash @_i\textsc{exists}(@_j c) \to \phi$ *then* $\vdash \phi$, where i is a nominal distinct from j not occurring in ϕ. Completeness follows directly from Theorem 9.

8. Conclusion

Kripke semantics showed the utility of viewing modal operators as expressing classical quantification over accessible worlds. Our goal in this paper has been to present hybrid logic as a natural continuation of the Kripkean strategy, a continuation that pays technical dividends with regards to axiomatisation.

The primary point we have made is that hybridisation permits us to use the standard Henkin-style model construction to prove simple and general completeness results, even in very weak logics; in particular, we showed that the Henkin strategy can be used with the basic hybrid language $\mathcal{H}(@)$. To make the Henkin strategy work in such a weak language, non-orthodox rules are needed, and indeed they are needed for two distinct purposes. Firstly, we need the Name and BG rules (or variants such as the Paste$_\Diamond$ rule) to enable us to prove a Lindenbaum Lemma strong enough to support a Henkin-style construction in which worlds are built out of (equivalence classes of) nominals. Secondly, to capture the logics of frame classes that cannot be axiomatised by pure axioms, we need existential saturation rules. We argued that both styles of rule, though unorthodox, are natural. Rules such as Name, BG, and Paste$_\Diamond$ directly reflect proof-theoretical ideas encountered in tableaux, natural deduction, or sequent systems for hybrid logic. Existential saturation rules directly reflect the meaning of (pure, nominal free) $\forall\exists$-prenex sentences of $\mathcal{H}(@,\forall)$.

The ease with which the basic axiomatisation for $\mathcal{H}(@)$ extends to richer logics is further testimony to its naturalness. As we saw, it is straightforward to enrich the modal base language (among other things, extremely simple axiomatizations of tense logical systems and the universal modality are possible). Furthermore, we can work our way up the hybrid expressivity hierarchy, first to $\mathcal{H}(@,\downarrow)$, and then to $\mathcal{H}(@,\forall)$. As we move upwards, simple axiomatizations that build directly on the base provided by $\mathbf{K}_{\mathcal{H}(@)}$ can be given, or (at the cost of a few extra axiom schema) we can eliminate the use of Name and BG. We can also make use of existential saturation rules in these richer systems, though they have less work to do in $\mathcal{H}(@,\downarrow)$ and become completely redundant in $\mathcal{H}(@,\forall)$. Hybrid logic has long been regarded as providing a fine-grained *semantic* perspective on various fragments of first-order logic. In our view, the results proved here show that it also provides a fine-grained *deductive* perspective too. Moreover, this deductive perspective is robust: it remains useful even when we move from propositional hybrid logics to quantified hybrid logic. Among other things, our approach permits completeness results for novel conditions on domains to be proved with ease.

But we wish to close this paper on another note. Recently, two other general approaches to hybrid axiomatisation have been developed. First, it is possible to take a different perspective on the basic hybrid language, a perspective under which our first axiomatisation, namely $\mathbf{K}_{\mathcal{H}(@)}$ (with no additional axioms) can be viewed as a Sahlqvist axiomatisation; see [41]. Under this approach, instead of adding pure axioms to the base logic, we add further Sahlqvist axioms to deal with additional frame conditions. Now, the intriguing thing about the Sahlqvist approach is that the completeness proof method used is *not* the Henkin method, it is the traditional modal method of canonical models. And this difference reflects a real divergence: the two methods of axiomatising richer logics are *not* additive. Adding a Sahlqvist axiom and a pure axiom to the base logic need not result in a logic complete with respect to the frame class that these two formulas jointly define!

The other line of axiomatic work concerns results for frame conditions that are not first-order (Sahlqvist axioms and pure axioms always define elementary conditions on frames). Now, there are important non-elementary conditions on frames, and many richer modal languages (Propositional Dynamic Logic is a good example) make use of concepts (such as transitive closures of relations) which are not first-order definable. Results in [4, 36] show that, under certain conditions, completeness (as well as other properties such as complexity and interpolation) transfer automatically from such modal logics to the corresponding hybrid logics. This technique can be applied to obtain easy completeness results for a wide range of non-elementary hybrid logics. Once again though, it is far from clear when such approaches can be combined with the approach presented here.

Summing up, there are currently three approaches to hybrid axiomatization: the classic approach (which dates back to Bull's pioneering work) which we have explored and refined in this paper, an approach based on modal Sahlqvist theory, and an approach by transfer from the basic modal language. So we have three different (and, at least from our present state of knowledge) seemingly incompatible routes to hybrid completeness. A natural next step would be to try and determine the extent to which they can be integrated.

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