Chapter 11

Bisimulations

11.1 Introduction

Both transition structures and valued transition structures are examples of relational structures. There is a standard way of comparing the algebraic properties of two such structures, namely through the notion of a *morphism*.

11.1 DEFINITION. (a) A morphism

$$\mathcal{A} \xrightarrow{f} \mathcal{B}$$

from a structure $\mathcal{A}$ to a structure $\mathcal{B}$ of the same signature is a function

$$f : A \longrightarrow B$$

from the carrier A of $\mathcal{A}$ to the carrier B of $\mathcal{B}$ such that

(Rel$^{\rightarrow}$) For each label i and elements a, x of $\mathcal{A}$

$$a \xrightarrow{i} x \quad \Rightarrow \quad f(a) \xrightarrow{i} f(x)$$

holds.

(b) A morphism

$$(\mathcal{A}, \alpha) \xrightarrow{f} (\mathcal{B}, \beta)$$

from a valued structure $(\mathcal{A}, \alpha)$ to a valued structure $(\mathcal{B}, \beta)$ is a morphism f between unadorned structures (as in (a)) such that

(Val$^{\rightarrow}$) For each variable P and element a of $\mathcal{A}$

$$a \in \alpha(P) \quad \Rightarrow \quad f(a) \in \beta(P)$$

holds. ■

It is important to notice that these two defining conditions are *implications*, and not equivalences. All the standard universal algebraic constructions are carried out with these morphisms in mind, nevertheless there are situations where a more restricted notion of morphism is desirable. Modal logic is one of these cases.

Given a morphism f (as above) it is natural to consider which formulas are preserved or reflected by f, that is, the formulas ϕ such that for each element a of $\mathcal{A}$, one or other of the two implications

$$(\mathcal{A},\alpha,a) \Vdash \phi \quad \Leftrightarrow \quad (\mathcal{B},\beta,f(a)) \Vdash \phi \qquad (11.1)$$

holds. For a general morphism f the corresponding class of formulas ϕ is not very interesting, hence the need to restrict the morphisms in some way.

Suppose first that we are interested in those morphisms f for which the equivalence (11.1) holds for all formulas ϕ and elements a of $\mathcal{A}$. A rather simple restriction on f will ensure this, and the resulting morphisms are variously known as *zigzag morphism* or *p-morphisms* (depending on the literary pretensions of the author). What is interesting about this restriction is that, once it has been elucidated, it becomes clear that it is also applicable to relations (as well as functions) between A and B. Those relations restricted in this way are called *bisimulations*, and this chapter is devoted to a full discussion of these.

A more extensive problem is to look for morphisms f for which the equivalence (11.1) holds for a restricted class of formulas. In this rather general form nothing beyond rather superficial results can be expected (for there are just too many parameters involved). There is, however, a particularly distinguished class of such morphisms known as *filtrations*. These are discussed in the next chapter.

11.2 Zigzag morphisms

In the literature you will find the terms 'zigzag morphism' and 'p-morphism' used more or less interchangeably. However, on closer inspection you will also find there are two closely related notions involved and it is instructive to make a pedantic distinction between these. Thus here we will use the two terms for the following notions

11.2 DEFINITION. We need two more kinds of morphisms.

(a) A *p-morphism*

$$\mathcal{A} \xrightarrow{\ f\ } \mathcal{B}$$

between two structures $\mathcal{A}$ and $\mathcal{B}$ of the same signature is a morphism f such that

(Rel$^{\leftarrow}$) For each label i and elements a of $\mathcal{A}$ and y of $\mathcal{B}$ with $f(a) \xrightarrow{i} y$, there is an element x of $\mathcal{A}$ with $a \xrightarrow{i} x$ and $f(x) = y$.

holds.

(b) A *zigzag morphism*

$$(\mathcal{A},\alpha) \xrightarrow{\ f\ } (\mathcal{B},\beta)$$

between two valued structures is a p-morphism f (as in (a)) such that

(Val$^{\leftarrow}$) For each variable P and element a of $\mathcal{A}$ the implication

$$a \in \alpha(P) \quad \Leftarrow \quad f(a) \in \beta(P)$$

holds. ■

The condition (Val$^{\leftarrow}$) is the converse of the condition (Val$^{\rightarrow}$) of Section 11.1. Since f is a morphism it must also satisfy this condition so each zigzag morphism satisfies the equivalence

$$(\text{Val}^{\leftrightarrow}) \qquad a \in \alpha(P) \quad \Leftrightarrow \quad f(a) \in \beta(P)$$

(for all appropriate P and a). Note that given any p-morphism

$$\mathcal{A} \xrightarrow{\ f\ } \mathcal{B}$$

and valuation β on $\mathcal{B}$, the equivalence Val$^{\leftrightarrow}$ may be used to construct a valuation α on $\mathcal{A}$ such that f becomes a zigzag morphism

$$(\mathcal{A},\alpha) \xrightarrow{\ f\ } (\mathcal{B},\beta).$$

For this reason the distinction we have made between p-morphism and zigzag morphism need not be followed too enthusiastically.

The reason for introducing zigzag morphism is explained by the following result.

11.3 THEOREM. *Let*

$$(\mathcal{A},\alpha) \xrightarrow{\ f\ } (\mathcal{B},\beta)$$

be a zigzag morphism. Then

$$(\mathcal{A},\alpha,a) \Vdash \phi \quad \Leftrightarrow \quad (\mathcal{B},\beta,b) \Vdash \phi$$

holds for all elements a of $\mathcal{A}$ and formulas ϕ.

We need not prove this result here since it is a simple consequence of the broader analysis which is the main topic of this chapter.

11.3 Bisimulations

Given a pair

$$(\mathcal{A},\alpha) \quad , \quad (\mathcal{B},\beta)$$

of valued structures (of the same signature) we we interested in relations

$$R \subseteq A \times B$$

which connect the semantical properties of the structures. There are two such relations of particular interest.

11.4 DEFINITION. Consider two valued structures (as above).

($\sim$) Let $\sim$ be the relation between A and B defined by

$$a \sim b \quad \Leftrightarrow \quad (\forall P \in Var)[a \in \alpha(P) \Leftrightarrow b \in \beta(P)]$$

for $a \in A$ and $b \in B$. We say an arbitrary relation R is a *matching* if $R \subseteq\sim$.

($\approx$) Let $\approx$ be the relation between A and B such that for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$

$$a \approx b$$

holds precisely when the equivalence

$$(\mathcal{A},\alpha,a) \Vdash \phi \quad \Leftrightarrow \quad (\mathcal{B},\beta,b) \Vdash \phi$$

holds for all formulas ϕ. We call $\approx$ the *semantic equivalence* relation. ■

Note that the semantic equivalence relation $\approx$ is a matching. Our objective is to look for various approximations to $\approx$.

11.5 DEFINITION. Consider two valued structures (as above).

(a) A relation R has the *back and forth* property if for each label i and elements a of $\mathcal{A}$ and b of $\mathcal{B}$, if aRb then both

back $\quad (\forall y \prec_i b)(\exists x \prec_i a)[xRy]$

forth $\quad (\forall x \prec_i a)(\exists y \prec_i b)[xRy]$

hold.

(b) A *bisimulation* is a matching which has the back and forth property. ■

For example, the empty relation is a bisimulation (vacuously), however we will see more interesting examples later. Before we do that let us see how $\approx$ and bisimulations are connected.

11.6 THEOREM. *Each bisimulation between two valued structures $(\mathcal{A},\alpha)$ and $(\mathcal{B},\beta)$ is included in the semantic equivalence relation.*

Proof. Let R be the given bisimulation. For each formula ϕ consider the following condition

(ϕ) $\quad$ For each $a \in A$ and $b \in B$, if aRb then, $\quad a \Vdash \phi \Leftrightarrow b \Vdash \phi$.

We show that (ϕ) holds for all formulas ϕ by induction on the complexity of ϕ.

The base case holds since R is a matching, and the passage across the boolean connectives is immediate. Thus it suffices to consider the passage across $[i]$ for an arbitrary label i.

Consider any formula $[i]\phi$ and elements a and b with aRb . Then

$$a \Vdash [i]\phi \Leftrightarrow (\forall x \prec_i a)[x \Vdash \phi] \Leftrightarrow (\forall y \prec_i b)[y \Vdash \phi] \Leftrightarrow b \Vdash [i]\phi$$

where the central equivalence is verified using the Induction Hypothesis and the back and forth property, as follows.

Suppose $x \Vdash \phi$ for each $x \prec_i a$ and consider any $y \prec_i b$. The back condition gives us a particular $x \prec_i a$ with xRy. But then (ϕ) gives

$$x \Vdash \phi \quad \Leftrightarrow \quad y \Vdash \phi$$

and hence $y \Vdash \phi$. This proves one half of the equivalence and the converse follows in a similar fashion.

This completes all the required induction steps. ■

This result shows that the semantic equivalence relation $\approx$ sits below $\sim$ and above all bisimulations. We need to close this gap, but before we do that let us see how bisimulations subsume zigzag morphisms.

Each morphism

$$\mathcal{A} \xrightarrow{\ f\ } \mathcal{B}$$

gives us a relation $F \subset A \times B$, namely its graph defined by

$$aFb \quad \Leftrightarrow \quad f(a) = b$$

(for $a \in A$ and $b \in B$).

11.7 THEOREM. *Consider any morphism f with graph F, as above.*

(a) This morphism f is a p-morphism precisely when F has the back and forth property

(b) Given valuations on the structures, the morphism f is a zigzag morphism precisely when the F is a bisimulation.

Proof. We will actually prove something a bit more detailed. Thus consider any function

$$f : A \longrightarrow B.$$

We show the following.

(i) The function f has $\mathrm{Rel}^{\rightarrow}$ (and hence is a morphism) precisely when F has the forth property.

(ii) The function f has $\mathrm{Rel}^{\leftarrow}$ precisely when F has the back property.

(iii) Given valuations α on $\mathcal{A}$ and β on $\mathcal{B}$, the function f has $\mathrm{Val}^{\leftrightarrow}$ precisely when F is a matching.

We prove these three correspondences in turn.

(i) Suppose first that f has $\mathrm{Rel}^{\rightarrow}$ and consider any $a, x \in A$ and $b \in B$ with

$$aFb \quad , \quad a \xrightarrow{i} x.$$

Then $f(a) = b$ so that, by $\mathrm{Rel}^{\rightarrow}$, we have $b \xrightarrow{i} f(x)$, and hence we may set $y = f(x)$ to verify the forth property. Conversely, suppose that F has the forth property, and consider any $a, x \in A$ with $a \xrightarrow{i} x$. Setting $b = f(x)$ we have aFb so that (by the forth property) there is some $y \in B$ with

$$xFy \quad , \quad b \xrightarrow{i} y.$$

Since $y = f(x)$, this gives $f(a) \xrightarrow{i} f(x)$, as required.

(ii) Suppose first that f has $\mathrm{Rel}^{\leftarrow}$ and consider any $a \in A$ and $b, y \in B$ with

$$aFb \quad , \quad b \xrightarrow{i} y.$$

Then $f(a) = b$ so that $f(a) \xrightarrow{i} y$ and hence, by $\mathrm{Rel}^{\leftarrow}$ there is some $x \in A$ with

$$f(x) = y \quad , \quad a \xrightarrow{i} x.$$

This verifies the back property. The converse implication follows in the same way.

(iii) This is immediate. ■

This result when combined with Theorem 11.6 gives a proof of Theorem 11.3.

We will say nothing more about zigzag morphisms; all we need to know about them can be deduced as the functional case of a bisimulation.

11.4 The largest bisimulation

For the remainder of the chapter let

$$(\mathcal{A}, \alpha) \quad , \quad (\mathcal{B}, \beta)$$

be a fixed pair of valued structures. We know there is at least one bisimulation between this pair, namely the empty relation. This, however, does not hold the attention for very long (even though it may be the only bisimulation). A more interesting example is at the opposite end of the scale.

11.8 THEOREM. *There is a unique largest bisimulation i.e. a bisimulation which includes all other bisimulations.*

Proof. Let $\mathcal{R}$ be the family of all bisimulations. We know that $\mathcal{R}$ is non-empty (since $\emptyset \in \mathcal{R}$). Set

$$S = \bigcup \mathcal{R}$$

i.e. let S be the relation such that

$$aSb$$

holds (for $a \in A$ and $b \in B$) precisely when there is some $R \in \mathcal{R}$ with aRb. Clearly S is a matching (and, in fact, it is included in $\approx$) and includes all bisimulations. Thus it suffices to show that S has the back and forth property.

For a fixed label i consider any $a \in A$ and $b \in B$. Consider also any $x \in A$ with $x \prec_i a$. By definition of S there is some $R \in \mathcal{R}$ with aRb. But then, since this R is a bisimulation, there is some $y \prec_i b$ with xRy. In particular, xSy, and so we have verified the forth property. The back property is verified in the same way. ■

Because of its special position we let

$$\approxeq$$

be this largest bisimulation. In particular we have

$$\approxeq \ \subseteq \ \approx \ \subseteq \ \sim$$

which gives us lower and upper bounds for $\approx$. In general these three relations are distinct, but there is an interesting situation where the two lower ones agree.

We say a structure $\mathcal{A}$ is *image finite* if for each label i and element a of $\mathcal{A}$, there are just finitely many elements x with

$$a \xrightarrow{i} x.$$

In particular, every finite structure and every deterministic structure is image finite.

11.9 **THEOREM.** *Suppose both the structures $\mathcal{A}$ and $\mathcal{B}$ are image finite. Then the two relations $\approxeq$ and $\approx$ coincide.*

Proof. It suffices to show that $\approx$ has the back and forth property. We will verify the forth property; the verification of the back property then follows in a similar fashion.

Consider elements $x, a \in A$ and $b \in B$ with $a \approx b$ and $x \prec_i a$ (for some label i). We must produce some $y \in B$ with $y \prec_i b$ and $x \approx y$.

Let $y_1, \dots, y_n$ be all the elements $y \in B$ with $y \prec_i b$. We may restrict our search to this finite set. By way of contradiction suppose there is no such y_r with $x \approx y_r$. Then for each $1 \le r \le n$, there is a formula θ_r with

$$x \Vdash \neg\theta_r \quad , \quad y_r \Vdash \theta_r.$$

Let ϕ be

$$\theta_1 \vee \ldots \vee \theta_n$$

so that $y_r \Vdash \phi$ for each $1 \le r \le n$, and hence

$$b \Vdash \Box\phi.$$

But then, since $a \approx b$, we have

$$a \Vdash \Box\phi$$

and hence $x \Vdash \phi$ which produces some $1 \le r \le n$ with

$$x \Vdash \theta_r.$$

Since this is a direct contradiction, the proof is complete. ■

11.5 A hierarchy of matchings

The construction of $\approxeq$ given in the proof of Theorem 11.8 does not provide much information beyond that $\approxeq$ is a bisimulation in a special position. There is another construction of $\approxeq$ (this time from above rather than below) which also provides a measure of its complexity. To describe this we need some preliminaries.

Given a relation R, its *derivative* is the relation $R^\blacktriangledown$ defined such that, for each $a \in A$ and $b \in B$,

$$aR^\blacktriangledown b$$

holds precisely when

$$aRb$$

and for each label i both

- $(\forall x \prec a)(\exists y \prec_i b)[xRy]$
- $(\forall y \prec b)(\exists x \prec_i a)[xRy]$

hold. Trivially $R^\blacktriangledown \subseteq R$. Note also that the operation $(\cdot)^\blacktriangledown$ is monotone, i.e.

$$S \subseteq R \quad \Rightarrow \quad S^\blacktriangledown \subseteq R^\blacktriangledown$$

holds for all relations R and S.

Bisimulations are precisely those matchings which are the fixed points of $(\cdot)^\blacktriangledown$, i.e. matchings R such that $R^\blacktriangledown = R$. This enables us to apply a standard procedure for obtaining fixed points.

Let *Ord* be the class of ordinals. For a given relation R we define the descending chain

$$(R_\alpha \mid \alpha \in Ord)$$

by an iteration of $(\cdot)^\blacktriangledown$. Thus we set

$$R_0 = R \quad , \quad R_{\alpha+1} = (R_\alpha)^\blacktriangledown \quad , \quad R_\lambda = \bigcap\{R_\alpha \mid \alpha < \lambda\}$$

for each ordinal α and limit ordinal λ. On cardinality grounds this chain eventually stabilizes, i.e. there is some ordinal ∞ such that

$$R_\alpha = R_\infty$$

for all ordinals $\alpha \ge \infty$. In fact, ∞ is the first ordinal α such that $R_{\alpha+1} = R_\alpha$. In particular, if the parent relation R is a matching, then the stable descendant R_∞ is a bisimulation.

11.10 **THEOREM.** *For each matching R, the matching R_∞ is the largest bisimulation included in R.*

Proof. We have already noted that R_∞ is a bisimulation and $R_\infty \subseteq R$. Consider any other bisimulation $S \subseteq R$. Then, using the monotone property of $(\cdot)^\blacktriangledown$ we have

$$S = S^\blacktriangledown \subseteq R^\blacktriangledown = R_1.$$

In the same way an obvious induction shows that

$$S \subseteq R_\alpha$$

for all $\alpha \in Ord$; in particular, $S \subseteq R_\infty$, as required. ■

Since $\approxeq$ is the largest bisimulation and is included in $\sim$, it is the largest bisimulation included in $\sim$. Thus we may apply the above construction to obtain $\approxeq$ from $\sim$. We let

$$\sim_0 \;=\; \sim \quad , \quad \sim_{\alpha+1} \;=\; (\sim_\alpha)^\blacktriangledown \quad , \quad \sim_\lambda \;=\; \bigcap\{\sim_\alpha \mid \alpha < \lambda\}$$

for each ordinal α and limit ordinal λ. Then $\approxeq$ is just $\sim_\infty$. This ordinal ∞ is a measure of the distance between $\sim$ and $\approxeq$, and hence tells us something about the complexity of $\approxeq$.

11.6 An example

At this stage in the proceedings it is instructive to look at a class of (particularly simple) examples. These examples show that the required value of ∞ can be indefinitely large.

In all the examples the two valued structures $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \beta)$ are the same, so we may concentrate on just one of them, $\mathcal{A}$. Also, for each variable P, we set

$$\alpha(P) = A.$$

This means that the relation $\sim$ is just $A \times A$, i.e. that $a \sim b$ holds for all $a, b \in A$.

The structure $\mathcal{A}$ is monomodal, in fact

$$\mathcal{A} = (A, \longrightarrow)$$

where A is an ordinal and, for $a, b \in A$

$$a \longrightarrow b \quad \Leftrightarrow \quad b < a$$

(where $<$ is the standard ordering on A).

For these examples we have a concise description of the relations $\sim_\alpha$.

11.11 PROPOSITION. *For each ordinal α and $a, b \in A$, the conditions*

(i) $a \sim_\alpha b$
(ii) $a = b$ *or* $\alpha \leq a, b$

are equivalent.

Proof. This is proved by induction on α.

The base case $\alpha = 0$ is trivial since $a \sim_0 b$ and $0 \leq a, b$ hold for all a, b.

For the induction step $\alpha \mapsto \alpha + 1$, suppose first that $a \sim_{\alpha+1} b$ and that $a \neq b$. Then $a < b$ (say) so, setting $y = a$ the back property of $\sim_{\alpha+1} = (\sim_\alpha)^\blacktriangledown$ produces some $x < a$ with $x \sim_\alpha y$. Since $x \neq y$ the Induction Hypothesis gives $\alpha \leq x, a$, so that

$$\alpha \leq x < a < b$$

and hence $\alpha + 1 \leq a, b$.

Conversely, suppose that $\alpha + 1 \leq a, b$ and consider any $x < a$, We require some y, b with $x \sim_\alpha y$. But if $x > a$ then we may take $y = x$, and if $\alpha \leq x$ then we may take $y = \alpha$, for in both cases the Induction Hypothesis gives $x \sim_\alpha y$.

For the induction leap to a limit ordinal λ we argue

$$\begin{aligned} a \sim_\lambda b &\Leftrightarrow (\forall \alpha < \lambda)[a \sim_\alpha b] \\ &\Leftrightarrow (\forall \alpha < \lambda)[a = b \text{ or } \alpha \leq a, b] \\ &\Leftrightarrow a = b \text{ or } (\forall \alpha < \lambda)[\alpha \leq a, b] \quad \Leftrightarrow \quad a = b \text{ or } \lambda \leq a, b \end{aligned}$$

which is the required result. ■

This class of examples shows that for each ordinal α the two relations $\sim_\alpha$ and $\sim_{\alpha+1}$ can be distinct. For let A be an ordinal which is at least $\alpha + 2$. Then

$$a = \alpha \quad , \quad b = \alpha + 1$$

are both members of A, and clearly

$$a \sim_\alpha b \quad , \quad \text{not}[a \sim_{\alpha+1} b].$$

which witnesses the required distinctness.

11.7 Stratified semantic equivalence

We now return to considering the fixed pair

$$(\mathcal{A}, \alpha) \quad , \quad (\mathcal{B}, \beta)$$

of valued structures. Each set of formulas Γ gives us a relation $|\Gamma|$ between these structures where, by definition,

$$a \mid \Gamma \mid b$$

holds (for $a \in A$ and $b \in B$) precisely when

$$(\forall \phi \in \Gamma)[a \Vdash \phi \quad \Leftrightarrow \quad b \Vdash \phi].$$

Thus, for example,

$$|\, Var \,| \text{ is } \sim \qquad \text{and} \qquad |\, Form \,| \text{ is } \approx .$$

Notice that many different sets Γ can give the same relation $|\Gamma|$. In particular, if Γ^B is the boolean closure of Γ, then

$$|\Gamma^B| \quad = \quad |\Gamma|.$$

These relations $|\Gamma|$ are most useful when Γ is closed under subformulas, but we need not assume this in general.

For each set Γ let $\Gamma^\Box$ be the set of formulas of the form

$$\phi \quad \text{or} \quad [i]\phi$$

for some $\phi \in \Gamma$ and label i. We are interested in the comparison between $|\Gamma|$ and $|\Gamma^\Box|$.

11.12 LEMMA. *For each set of formulas Γ, we have $|\Gamma|^\blacktriangledown \subseteq |\Gamma^\Box|$.*

Proof. The argument for this is essentially the same as the argument verifying the induction step in the proof of Theorem 11.6. ■

Next let

$$\Delta_0 = Var \cup \{\top, \bot\}$$

so that $(\Delta_0)^B$ is the set of Box-free (i.e. propositional) formulas and hence (using the above remark)

$$|\,\Delta_0\,| \;=\; \sim \;=\; \sim_0 .$$

Now for each $r < \omega$ set

$$\Delta_{r+1} = (\Delta_r^B)^\Box$$

to produce an ascending chain

$$\Delta_0 \subseteq \Delta_0^B \subseteq \cdots \subseteq \Delta_r \subseteq \Delta_r^B \subseteq \Delta_{r+1} \subseteq \cdots$$

with

$$\Delta_\omega = \bigcup\{\Delta_r \mid r < \omega\} = Form$$

and hence

$$|\,\Delta_\omega\,| \;=\; \approx .$$

We are interested in the intermediate relations $|\,\Delta_r\,|$.

11.13 LEMMA. *For each $r < \omega$, we have* $\sim_r \;\subseteq\; |\,\Delta_r\,|$.

Proof. We have seen already that

$$\sim_0 \;=\; |\,\Delta_0\,| .$$

Now assuming the inclusion holds for r, we have

$$\sim_{r+1} \;=\; \sim_r^\blacktriangledown \;\subseteq\; |\,\Delta_r\,|^\blacktriangledown \;=\; |\,\Delta_r^B\,|^\blacktriangledown \;\subseteq\; |\,\Delta_r^{B\Box}\,| \;=\; |\,\Delta_{r+1}\,|$$

where the crucial step (the second inclusion) follows by Lemma 11.12.
The required result now follows by induction. ■

Taking the limiting case of this gives the following.

11.14 COROLLARY. $\sim_\omega \;\subseteq\; \approx$.

We have seen already a condition which ensures that the two relations $\sim_\omega$ and $\approx$ coincide, namely image finiteness. But then $\approx$ also coincides with $\approxeq$. It is, therefore, of interest to find natural conditions which force $\sim_\omega$ and $\approx$ to agree but where these relations may still be distinct from $\approxeq$.

To this end, let us say a set of formulas Γ is *essentially finite* for the pair of valued structures in question, if there are finitely many members $\gamma_1, \ldots, \gamma_n$ of Γ such that for each $\phi \in \Gamma$ there is such a γ with

$$(\mathcal{A}, \alpha), (\mathcal{B}, \beta) \Vdash^v (\phi \leftrightarrow \gamma).$$

For example, any finite set is essentially finite. Also the set *Var* is essentially finite for the ordinal example of Section 11.6.

Recall that we say the signature is finite if there are just finitely many labels.

11.15 LEMMA. *Suppose the signature is finite and let Γ be an essentially finite set of formulas. Then Γ^B and $\Gamma^\Box$ are also essentially finite.*

Proof. Let $\gamma_1, \ldots, \gamma_n$ be the given finitely many members of Γ which span Γ. We are interested in the formulas ψ of the form

$$\pm\gamma_1 \wedge \ldots \wedge \pm\gamma_n$$

where each $\pm\gamma$ is either γ or $\neg\gamma$. Notice that there are just finitely many such formulas ψ.

Each boolean combination of $\gamma_1, \ldots, \gamma_n$ is tautologically equivalent to a formula ϕ of the shape

$$\psi_1 \vee \ldots \vee \psi_m$$

where each formula ψ_r is a formula of the kind ψ just described. Thus we see that Γ^B is essentially finite.

Finally, since the set of labels is finite, we see that $\Gamma^\Box$ is essentially finite. ■

The next result explains why we have introduced this notion of essential finiteness.

11.16 PROPOSITION. *If the set of formulas Γ is essentially finite then the two relations $|\,\Gamma\,|^\blacktriangledown$ and $(\Gamma^\Box)$ coincide.*

Proof. By Lemma 11.12 we know already that

$$|\,\Gamma\,|^\blacktriangledown \;\subseteq\; (\Gamma^\Box).$$

Thus it suffices to show the converse inclusion, assuming that Γ is essentially finite.

Let $\gamma_1, \ldots, \gamma_n$ be the given spanning members of Γ. Consider any $a \in A$ and $b \in B$ with $a \mid \Gamma^\Box \mid b$. Consider also any $x \prec_i a$ (for some label i). We must produce some $y \prec_i b$ with $x \mid \Gamma \mid y$.

Set

$$\psi \;:=\; \pm\gamma_1 \wedge \ldots \wedge \pm\gamma_n$$

where each $\pm\gamma$ is γ or $\neg\gamma$ with the sign chosen so that

$$x \Vdash \pm\gamma.$$

Note that

$$x \Vdash \psi \quad , \quad \psi \in \Gamma^{\Box} \quad , \quad [i]\neg\psi \in \Gamma^{\Box}.$$

Now $a \Vdash \langle i \rangle\psi$ so that $b \Vdash \langle i \rangle\psi$ (since $a \mid \Gamma^{\Box} \mid b$) which gives some $y \prec_i b$ with $y \Vdash \psi$. It suffices to show that $x \mid \Gamma \mid y$. But for each $\phi \in \Gamma$ there is some γ with

$$x \Vdash (\phi \leftrightarrow \gamma) \quad , \quad y \Vdash (\phi \leftrightarrow \gamma)$$

and hence

$$x \Vdash \phi \quad \Leftrightarrow \quad y \Vdash \phi$$

by the relationship of x and y to ψ. ■

Finally we have arrived at the result we have been travelling towards.

11.17 THEOREM. *Suppose that the signature is finite and that Var is essentially finite. Then for each $r < \omega$ the two relations $\sim_r$ and $\mid \Delta_r \mid$ agree. In particular $\sim_\omega$ and $\approx$ coincide.*

Proof. By definition, $\sim_0$ and $\mid \Delta_0 \mid$ agree. Also, assuming that $\sim_r$ and $\mid \Delta_r \mid$ agree, an application of Proposition 11.16 gives

$$\sim^{r+1} \quad = \quad (\sim_r)^{\blacktriangledown} \quad = \quad (\mid \Delta_r \mid)^{\blacktriangledown} \quad = \quad \mid (\Delta_r)^{\Box} \mid \quad = \quad (\mid \Delta_{r+1} \mid)$$

so the required result follows by induction. ■

This result shows that for the ordinal examples of Section 11.6 the two relations $\sim_\omega$ and $\approx$ coincide. Notice also that for these examples the relation $\approxeq$ can be much smaller.

11.8 Exercises

11.1 Structures of the various kinds form several categories.

(a) Show that the composite of two morphisms (between structures of the same kind) is itself a morphism.

(b) Show that the composite of two p-morphisms is itself a p-morphism, and the composite of two zigzag morphisms is a zigzag morphism.

11.2 For the monomodal language, the singleton set $\{0\}$ carries just two transition structures

$$\mathcal{L} \quad , \quad \mathcal{R}$$

where the transition relation on $\mathcal{L}$ is empty and on $\mathcal{R}$ it is not.

(a) Show that the unique assignment $g : A \longrightarrow \{0\}$ defines a morphism

$$\mathcal{A} \xrightarrow{g} \mathcal{R}$$

and that this morphism is a p-morphism precisely when $\mathcal{A}$ is serial.

(b) Show that the morphisms

$$\mathcal{L} \xrightarrow{f} \mathcal{A}$$

are in bijective correspondence with the elements of $\mathcal{A}$, and that the p-morphisms (in this direction) are in bijective correspondence with the blind elements of $\mathcal{A}$.

(c) Determine the morphisms

$$\mathcal{A} \longrightarrow \mathcal{L} \quad , \quad \mathcal{R} \longrightarrow \mathcal{A}.$$

Which of these are p-morphisms?

11.3 Let

$$\mathcal{A} \xrightarrow{f} \mathcal{B}$$

be a p-morphism and let α be a valuation on $\mathcal{A}$. Show there are two valuations λ and ρ on $\mathcal{B}$ such that the following hold.

(a) Both

$$(\mathcal{A}, \alpha) \xrightarrow{f} (\mathcal{B}, \lambda) \quad , \quad (\mathcal{A}, \alpha) \xrightarrow{f} (\mathcal{B}, \rho)$$

are zigzag morphisms;

(b) For each valuation β on $\mathcal{B}$,

$$(\mathcal{A}, \alpha) \xrightarrow{f} (\mathcal{B}, \beta)$$

is a zigzag morphism if and only if $\lambda \leq \beta \leq \rho$, i.e.

$$\lambda(P) \subseteq \beta(P) \subseteq \rho(P)$$

holds for all variables P.

11.4 Let $\mathcal{A}$ and $\mathcal{B}$ be structures with $\mathcal{A} \subseteq \mathcal{B}$, and let f be the insertion $f : A \longrightarrow B$.

(a) Show that f is a morphism.

(b) Show that $\mathcal{A} \subseteq_g \mathcal{B}$ (in the sense of Exercise 4.9) if and only f is a p-morphism.