11.5 Let

$$R \subseteq A \times B \quad , \quad S \subseteq B \times C$$

be two relations. The sequential composite

$$R;S \subseteq A \times C$$

is the relation defined by

$$a(R;S)c \Leftrightarrow (\exists b \in B)[aRbSc]$$

(for $a \in A$ and $c \in C$).

(a) Show that the sequential composite of two back and forth relations is a back and forth relation, and the sequential composite of two bisimulations is a bisimulation.

(b) Let

$$f : A \longrightarrow B \quad , \quad g : B \longrightarrow C$$

be two functions with graphs F and G. Show that the sequential composite $F;G$ is the graph of the function composite gf.

11.6 Consider the structure

$$\mathcal{N} = (\mathsf{N}, \longrightarrow)$$

carried by the set of natural numbers N with the successor relation given by

$$a \longrightarrow b \Leftrightarrow a = b + 1$$

(for $a, b \in \mathsf{N}$). Let ν be the valuation on $\mathcal{N}$ with $\nu(P) = \mathsf{N}$ for all variables P, and consider the induced matching hierarchy $\sim_\bullet$.

(a) Show that for each $a, b \in \mathsf{N}$ and $r < \omega$

$$a \sim_{r+1} b \Leftrightarrow \begin{cases} a = b \le r \\ \text{or} \\ a, b > r. \end{cases}$$

Hence show that $\sim_\omega = \sim_\infty$ is just equality.

(b) For each $k \in \mathsf{N}$, find a sentence ϕ_k such that

$$a \Vdash \phi_k \Leftrightarrow a = k$$

holds for all $a \in \mathsf{N}$.

Chapter 12

Filtrations

12.1 Introduction

In Chapter 11 we isolated a class of valued morphisms

$$(\mathcal{A}, \alpha) \longrightarrow (\mathcal{B}, \beta)$$

namely the zigzag morphisms, for which the equivalence

$$(\mathcal{A}, \alpha, a) \Vdash \phi \quad \Leftrightarrow \quad (\mathcal{B}, \beta, f(a)) \Vdash \phi$$

holds for all elements $a \in A$ and *all* formulas ϕ. These morphisms are most useful when the two structures are are given independently and when we really are concerned with all formulas. However, there are many situations where we are given only the valued structure $(\mathcal{A}, \alpha)$ and we are required to construct a valued structure $(\mathcal{B}, \beta)$ together with an appropriate morphism f such that the equivalence holds only for a restricted class of formulas. More often than not we also require $\mathcal{B}$ to be finite and as small as possible. In this chapter we will look at the commonest method of obtaining such morphisms, namely the method of filtrations. As we will see later in Chapter 13 this method has some significant consequences for the completeness and decidability of various formal systems.

Throughout the chapter Γ is some fixed set of formulas which is assumed to be closed under subformulas. In most applications this set Γ is the set of all subformulas of a given formula. Our objective is to ensure that the equivalence holds for all formulas $\phi \in \Gamma$.

12.1 DEFINITION. Let $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \beta)$ be two valued structures.

(a) A Γ-*morphism* from the first to the second is a morphism

$$\mathcal{A} \xrightarrow{f} \mathcal{B}$$

such that for each variable $P \in \Gamma$

$$a \in \alpha(P) \quad \Rightarrow \quad f(a) \in \beta(P)$$

holds for all $a \in A$.

(b) A Γ-*filtration* from the first structure to the second structure is a Γ-morphism f, as above, such that:

(Sur) This morphism is surjective.

(Var) For each variable $P \in \Gamma$ and element $a \in A$

$$a \in \alpha(P) \quad \Leftrightarrow \quad f(a) \in \beta(P).$$

(Fil) For each label i and formula ϕ with $[i]\phi \in \Gamma$, the implication

$$a \Vdash [i]\phi \quad \Rightarrow \quad x \Vdash \phi$$

holds for all elements a, x of $\mathcal{A}$ with $f(a) \xrightarrow{i} f(x)$ in $\mathcal{B}$. ■

Notice that there are no restrictions at all on the variables $P \notin \Gamma$, in particular the filtration f need not be a full valued morphism.

The first thing to do now is to demonstrate why filtrations are useful.

12.2 **THEOREM.** *Let f be a Γ-filtration (as above). Then, for each formula $\phi \in \Gamma$ the equivalence*

$$(\mathcal{A}, \alpha, a) \Vdash \phi \quad \Leftrightarrow \quad (\mathcal{B}, \beta, f(a)) \Vdash \phi$$

holds for all elements a of $\mathcal{A}$.

Proof. We proceed by induction on ϕ.

The two base cases $\phi = \top$ and $\phi = \bot$ hold trivially. For the case $\phi = P \in Var$; either $P \in \Gamma$ in which case the equivalence is given by (Var), or $P \notin \Gamma$ in which case there is nothing to prove. The induction steps across propositional connectives follow easily since Γ is closed under subformulas, thus it remains to deal with the induction step across a box.

Fix a label i and consider any formula ϕ with $[i]\phi \in \Gamma$. Since we also have $\phi \in \Gamma$ the Induction Hypothesis gives

$$x \Vdash \phi \quad \Leftrightarrow \quad f(x) \Vdash \phi$$

for all $x \in A$. We must show that

$$a \Vdash [i]\phi \quad \Leftrightarrow \quad f(a) \Vdash [i]\phi$$

for all $a \in A$.

Suppose first that $a \Vdash [i]\phi$ and consider any $y \in B$ with $f(a) \xrightarrow{i} y$. Since f is surjective there is some $x \in A$ with $f(x) = y$. But then (Fil) gives $x \Vdash \phi$ and the Induction Hypothesis gives $y \Vdash \phi$, so that $f(a) \Vdash [i]\phi$. Conversely, suppose that $f(a) \Vdash [i]\phi$ and consider any $x \in A$ with $a \xrightarrow{i} x$. Since f is a morphism we have $f(a) \xrightarrow{i} f(x)$, so that $f(x) \Vdash \phi$ and hence the Induction Hypothesis gives $x \Vdash \phi$. Thus $a \Vdash [i]\phi$ as required. ■

In this proof I have been a little bit sloppy because I did not state precisely what the Induction Hypothesis is. Before you continue you should do this and make sure you understand the mechanism of the induction.

12.2 The canonical carrying set

We have already fixed the set Γ. We now fix the valued structure $(\mathcal{A}, \alpha)$ and turn to the central topic of this chapter.

> For the given $(\mathcal{A}, \alpha)$ and Γ, how can we construct a Γ-filtration of $(\mathcal{A}, \alpha)$ for which the target is as small as possible ?

Consider the equivalence relation $\sim$ on A (the carrying set of $\mathcal{A}$) defined by

$$a \sim x \quad \Leftrightarrow \quad (\forall \phi \in \Gamma)[a \Vdash \phi \Leftrightarrow x \Vdash \phi]$$

for all $a, x \in A$ and set

$$B = A/\!\sim$$

i.e. let B be the set of equivalence classes of A. For each $a \in A$ let $a^\sim$ be the equivalence class to which a belongs, i.e. let

$$a^\sim = \{x \in A \mid a \sim x\}.$$

Let

$$\begin{aligned} A &\xrightarrow{\;f\;} B \\ a &\longmapsto a^\sim \end{aligned}$$

be the canonical surjection. We wish to construct a valued structure $(\mathcal{B}, \beta)$ on B in such a way that f becomes a Γ-filtration.

Before we do this let us estimate the size of B.

12.3 **LEMMA.** *Suppose the set of formulas Γ is finite. Then so is B and, in fact, $card(B) \leq 2^{card(\Gamma)}$.*

Proof. The set

$$[\Gamma \to 2]$$

of all functions from Γ to 2 is finite with cardinality $2^{card(\Gamma)}$. For each $a \in \Gamma$ let

$$f_a : \Gamma \longrightarrow 2$$

be the function given by

$$f_a(\phi) = \begin{cases} 1 \text{ if } a \Vdash \phi \\ 0 \text{ if } a \Vdash \neg\phi \end{cases}$$

(for $\phi \in \Gamma$). The definition of $\sim$ can be rephrased as

$$a \sim x \quad \Leftrightarrow \quad f_a = f_x$$

in particular we have a well-defined injection

$$\begin{aligned} B &\longrightarrow [\Gamma \to 2] \\ a^{\sim} &\longmapsto f_a \end{aligned}$$

which gives the required result. ■

Our main problem is to convert B into a transition structure satisfying the appropriate conditions. It turns out that there are many ways of doing this, but there is a 'left-most' (or $\exists$-) solution and a 'right-most' (or $\forall$-) solution. We look at these two cases in detail.

12.3 The left-most filtration

For each label i let $\xrightarrow{i}_l$ be the relation on $B = A/{\sim}$ given by

$$b \xrightarrow{i}_l y \quad \Leftrightarrow \quad (\exists a \in b, x \in y)[a \xrightarrow{i} x]$$

(for $b, y \in B$). Let $\mathcal{B}^l$ be the corresponding structure on B. First a simple observation.

12.4 LEMMA. *The canonical assignment $f : A \longrightarrow B$ is a surjective morphism $\mathcal{A} \longrightarrow \mathcal{B}^l$.*

Proof. Suppose that $a \xrightarrow{i} x$ for some $a, x \in A$ and label i. Then

$$a \in f(a) \quad , \quad x \in f(x) \quad , \quad a \xrightarrow{i} x$$

and hence $a \xrightarrow{i}_l x$, as required. ■

Let λ be the valuation on $\mathcal{B}^l$ given by

$$b \in \lambda(P) \quad \Leftrightarrow \quad (\exists a \in b)[a \in \alpha(P)]$$

for all variables $P \in \Gamma$ and elements $a \in A$. The values $\lambda(P)$ for other variables P are not important, but for precision we may set

$$\lambda(P) = \emptyset$$

for such P.

12.5 THEOREM. *The assignment*

$$(\mathcal{A}, \alpha) \xrightarrow{f} (\mathcal{B}^l, \beta)$$

is a Γ-filtration.

Proof. By Lemma 12.4 and the construction of λ the assignment f is a surjective Γ-morphism from $(\mathcal{A}, \alpha)$ to $(\mathcal{B}^l, \lambda)$. It thus remains to verify properties (Var) and (Fil).

By construction we have

$$a \in \alpha(P) \quad \Rightarrow \quad f(a) \in \lambda(P)$$

for all $a \in A$ and variables $P \in \Gamma$. Conversely, if $f(a) \in \lambda(P)$ (for some $a \in A$ and $P \in \Gamma$) then, by definition of λ there is some $x \in f(a)$ with $x \in \alpha(P)$. But then $x \sim a$ so that, by definition of $\sim$, we have $a \in \alpha(P)$, as required to verify (Var).

Finally, to verify (Fil), consider any pair of elements $a, x \in A$ with $f(a) \xrightarrow{i}_l f(x)$ (in $\mathcal{B}^l$). Then, by definition of $\xrightarrow{i}_l$, there are $u, v \in A$ with

$$u \sim a \quad , \quad v \sim x \quad , \quad u \xrightarrow{i} v.$$

Hence, for each formula ϕ with $[i]\phi \in \Gamma$, the definition of $\sim$ gives

$$a \Vdash [i]\phi \Rightarrow u \Vdash [i]\phi \Rightarrow v \Vdash \phi \Rightarrow x \Vdash \phi$$

as required. ■

12.4 The right-most filtration

For each label i let $\xrightarrow{i}_r$ be the relation on $B = A/{\sim}$ given by

$$b \xrightarrow{i}_r y \Leftrightarrow \left\{ \begin{array}{c} \text{For each formula } \phi \text{ with } [i]\phi \in \Gamma, \\ (\forall a \in b, x \in y)[a \Vdash [i]\phi \Rightarrow x \Vdash \phi] \end{array} \right\}$$

(for $b, y \in B$). Let $\mathcal{B}^r$ be the corresponding structure on $\mathcal{B}$.

12.6 LEMMA. *The assignment*

$$f : A \longrightarrow B$$

is a surjective morphism $\mathcal{A} \longrightarrow \mathcal{B}^r$.

Proof. Suppose $a \xrightarrow{i} x$ and consider any $u \in f(a), v \in f(x)$. Then

$$u \sim a \quad , \quad x \sim v$$

and for each formula ϕ with $[i]\phi \in \Gamma$ we have

$$u \Vdash [i]\phi \Rightarrow a \Vdash [i]\phi \Rightarrow x \Vdash \phi \Rightarrow v \Vdash \phi$$

so that $f(a) \xrightarrow{i}_r f(x)$ as required. ■

Let ρ be the valuation on $\mathcal{B}^r$ given by

$$b \in \rho(P) \quad \Leftrightarrow \quad (\forall a \in b)[a \in \alpha(P)]$$

for all variables $P \in \Gamma$ and elements $a \in A$. The values $\lambda(P)$ for other variables P is not important, but for precision we may set

$$\lambda(P) = B$$

for such P.

12.7 THEOREM. *The assignment*

$$(\mathcal{A}, \alpha) \xrightarrow{f} (\mathcal{B}^l, \beta)$$

is a Γ*-filtration.*

Proof. We must verify the conditions (Var) and (Fil).

For (Var) consider any variable $P \in \Gamma$ and element $a \in \alpha(P)$. Then for each $x \in f(a)$, we have $x \sim a$, so that $x \in \alpha(P)$, and hence $f(a) \in \rho(P)$. Conversely, if $f(a) \in \rho(P)$ then, since $a \in f(a)$, we have $a \in \alpha(P)$ as required.

To verify (Fil) consider any $a, x \in A$ with $f(a) \xrightarrow{i}_r f(x)$. Then since $a \in f(a)$ and $x \in f(x)$, for each appropriate formula ϕ we have

$$a \Vdash [i]\phi \quad \Rightarrow \quad x \Vdash \phi$$

as required. ■

12.5 Filtrations sandwiched

So far we have at least two ways of converting the canonical quotient

$$A \xrightarrow{f} B$$

into a Γ-filtration. These are not the only possible constructions but, as we show in this section, all compatible Γ-filtration structures on B are sandwiched between $(\mathcal{B}^l, \lambda)$ and $(\mathcal{B}^r, \rho)$. Before we prove this, a simple observation.

12.8 LEMMA. *The identity map*

$$\begin{aligned} B &\longrightarrow B \\ b &\longmapsto b \end{aligned}$$

provides a Γ*-morphism* $(\mathcal{B}^l, \lambda) \longrightarrow (\mathcal{B}^r, \rho)$.

Proof. For an arbitrary label i consider any $b, y \in B$ with $b \xrightarrow{i}_l y$. By the definition of $\xrightarrow{i}_l$ there are $a \in b$ and $x \in y$ with $a \xrightarrow{i} x$. Now consider any $u \in b$ and $v \in y$. Then

$$u \sim a \quad , \quad x \sim v$$

so that, for each formula ϕ with $\Box\phi \in \Gamma$ we have

$$u \Vdash [i]\phi \Rightarrow a \Vdash [i]\phi \Rightarrow x \Vdash \phi \Rightarrow v \Vdash \phi$$

so that $b \xrightarrow{i}_r y$. This shows that the identity map is a morphism $\mathcal{B}^l \longrightarrow \mathcal{B}^r$.

Next consider any variable $P \in \Gamma$ and element $b \in B$. If $b \in \lambda(P)$ then, by definition of λ, there is some $a \in b$ with $a \in \alpha(P)$, and hence, by the definition of ρ, $b = f(a) \in \rho(P)$. Thus

$$b \in \lambda(P) \Rightarrow b \in \rho(P)$$

which is enough to complete the proof. ■

Consider now an arbitrary valued structure $(\mathcal{B}, \beta)$ based on the set $B = A/\sim$. We say $(\mathcal{B}, \beta)$ is Γ-*sandwiched* if both the following conditions hold.

- For each label i and elements $b, y \in B$ both the implications

$$b \xrightarrow{i}_l y \quad \Rightarrow \quad b \xrightarrow{i} y \quad \Rightarrow \quad b \xrightarrow{i}_r y$$

hold.

- For each variable $P \in \Gamma$ both the inclusions

$$\lambda(P) \subseteq \beta(P) \subseteq \rho(P)$$

hold.

These conditions delimit the range of Γ-filtrations.

12.9 THEOREM. *Let $(\mathcal{B},\beta)$ be a valued structure based on the quotient set B. Then the canonical assignment $f : A \longrightarrow B$ provides a Γ-filtration if and only if $(\mathcal{B},\beta)$ is Γ-sandwiched.*

Proof. Suppose first that f is a Γ-filtration.

Consider any label i and elements $b, y \in B$. If $b \xrightarrow{i}_l y$ then there are $a \in b$ and $x \in y$ with $a \xrightarrow{i} x$ hence, since f is a morphism

$$b = f(a) \xrightarrow{i} f(x) = y.$$

Also if $b \xrightarrow{i} y$ then for each $a \in b$ and $x \in y$ we have $f(a) \xrightarrow{i} f(x)$, so that (Fil) ensures that $b \xrightarrow{i}_r y$. Similarly, for each variable $P \in \Gamma$ and elements $a \in A$ and $b \in B$ with $f(a) = b$, we have

$$b \in \lambda(P) \Rightarrow a \in \alpha(P) \Rightarrow b \in \beta(P)$$

and

$$b \in \beta(P) \Rightarrow a \in \alpha(P) \Rightarrow b \in \rho(P)$$

as required.

This shows that $(\mathcal{B},\beta)$ is Γ-sandwiched.

Conversely, suppose we know that $(\mathcal{B},\beta)$ is Γ-sandwiched. Then, for each $a, x \in A$, since f is a morphism $\mathcal{A} \longrightarrow \mathcal{B}^l$, we have

$$a \xrightarrow{i} x \quad\Rightarrow\quad f(a) \xrightarrow{i}_l f(x) \quad\Rightarrow\quad f(a) \xrightarrow{i} f(x).$$

Thus f is a morphism $\mathcal{A} \longrightarrow \mathcal{B}$ and it remains to verify (Var) and (Fil).

The property (Var) follows almost immediately using both ends of the sandwich.

Finally, to verify (Fil), suppose that $f(a) \xrightarrow{i} f(x)$ (for some $a, x \in A$). Then $f(a) \xrightarrow{i}_r f(x)$ and hence, for each appropriate formula ϕ,

$$a \Vdash [i]\phi \quad\Rightarrow\quad x \Vdash \phi$$

as required. ■

12.6 Separated structures

Recall that in Definition 11.4 of Chapter 11 we introduced the semantic equivalence relation $\approx$ between two valued structures. As a particular case of this

we can consider the relation on one valued structure $(\mathcal{A},\alpha)$, i.e. the relation $\approx$ on A given by

$$a \approx x \Leftrightarrow \left\{ \begin{array}{c} \text{For all formulas } \phi, \\ (\mathcal{A},\alpha,a) \Vdash \phi \Leftrightarrow (\mathcal{A},\alpha,x) \Vdash \phi \end{array} \right\}$$

for all $a, x \in A$. We say a valued structure is *separated* if this relation is just equality, i.e. if

$$a \approx x \Rightarrow a = x$$

holds for all $a, x \in A$. In this section we observe how filtration constructions produce separated structures.

Thus, fix the usual data of a valued structure $(\mathcal{A},\alpha)$, a set of formulas Γ with the induced equivalence relation $\sim$ on A, and the quotient set $B = A/\!\sim$. Let

$$(\mathcal{A},\alpha) \xrightarrow{f} (\mathcal{B},\beta)$$

be a filtration where the target structure $(\mathcal{B},\beta)$ is carried by B. Note that

$$a \sim x \Leftrightarrow f(a) = f(x)$$

holds for all $a, x \in A$.

12.10 LEMMA. *In the circumstances given above, the structure $(\mathcal{B},\beta)$ is separated.*

Proof. For each $a, x \in A$ we have

$$\begin{array}{rcl} f(a) \approx f(x) & \Rightarrow & \left\{ \begin{array}{c} \text{For all formulas } \phi, \\ f(a) \Vdash \phi \Leftrightarrow f(x) \Vdash \phi \end{array} \right\} \\ & \Rightarrow & \left\{ \begin{array}{c} \text{For all formulas } \phi \in \Gamma, \\ f(a) \Vdash \phi \Leftrightarrow f(x) \Vdash \phi \end{array} \right\} \\ & \Rightarrow & \left\{ \begin{array}{c} \text{For all formulas } \phi \in \Gamma, \\ a \Vdash \phi \Leftrightarrow x \Vdash \phi \end{array} \right\} \\ & \Rightarrow & a \sim x \qquad \Rightarrow \quad f(a) = f(x) \end{array}$$

where these implications follow by

- the definition of $\approx$,
- restriction,

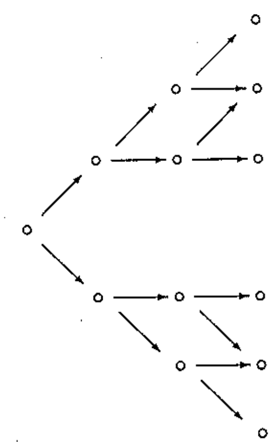
- the filtration preservation property,
- the definition of $\sim$,

and finally the above remark. ■

12.7 Exercises

12.1 Let Γ be the set of sentences (variable-free formulas). Note that for this set Γ the valuation plays very little part in the notion of a Γ-filtration.

Consider the 13 element monomodal structure $\mathcal{A}$



where all the transitions are displayed explicitly (in particular, no node is reflexive).

(a) Determine the equivalence relation $\sim$ on A induced by Γ.

(b) Show that the left-most and the right-most Γ-filtrations of $\mathcal{A}$ are identical and both have just four elements.

12.2 Let $\mathcal{N}^+$ and $\mathcal{N}^-$ be the two monomodal structures on $\mathbb{N}$ (the set of natural numbers) with transition relations given by

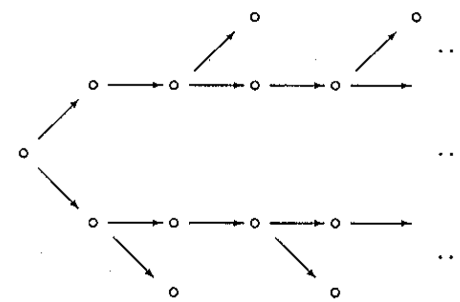
$$a \xrightarrow{+} b \Leftrightarrow b = a+1 \quad , \quad a \xrightarrow{-} b \Leftrightarrow a = b+1 .$$

Let Γ be the set of sentences.

(a) Show that the left-most Γ-filtration of $\mathcal{N}^+$ effects a complete collapse to a reflexive point.

(b) Show that the right-most Γ-filtration of $\mathcal{N}^-$ is an isomorphism.

12.3 Consider the infinite monomodal structure $\mathcal{A}$



where all the transition are displayed explicitly (in particular, no node is reflexive) Let Γ be the set of sentences.

Construct the left-most Γ-filtration of $\mathcal{A}$.

12.4 Show that the target of a filtration need be neither finite nor separated.

12.5 Let Γ be the set of variables. Let $\mathcal{A}$ be any monomodal structure and let α be any valuation on $\mathcal{A}$ such that for each $a \in A$ there is some $P \in \Gamma$ with $\alpha(P) = \{a\}$. Determine the left-most and right-most Γ-filtrations of $(\mathcal{A}, \alpha)$.

12.6 Suppose the valued structure $(\mathcal{A}, \alpha)$ is both finite and separated.

(a) Show that for each pair of distinct elements a, x of $\mathcal{A}$, there is a formula $\xi_{a,x}$ which distinguishes between a and x in the sense that both

$$(\mathcal{A}, \alpha, a) \Vdash \xi_{a,x} \quad \text{and} \quad (\mathcal{A}, \alpha, x) \Vdash \neg\xi_{a,x}$$

hold.

(b) Show that for each $a \in A$ there is a formula ρ_a such that

$$(\mathcal{A}, \alpha, x) \Vdash \rho_a \Leftrightarrow x = a$$

holds for all $x \in A$.

(c) Show that for each $X \subseteq A$ there is a formula τ_X such that

$$(\mathcal{A}, \alpha, a) \Vdash \tau_X \Leftrightarrow a \in X$$

holds for all $a \in A$.

Let μ be any valuation on $\mathcal{A}$ and let $(\cdot)^\mu$ be the substitution given by

$$P \longmapsto \tau_{\mu(P)}$$

for each variable P.