(d) Show that

$$(\mathcal{A},\mu,a) \Vdash \phi \Leftrightarrow (\mathcal{A},\alpha,a) \Vdash \phi^{\mu}$$

holds for all elements a of $\mathcal{A}$ and formulas ϕ.

12.7 Let Γ be the set of all formulas and let

$$(\mathcal{A},\alpha) \xrightarrow{f} (\mathcal{B},\beta)$$

be a surjective valued morphism.

(a) Show that if f is a zigzag morphism then f is a Γ-filtration.

(b) Show that if $(\mathcal{B},\beta)$ is finite and separated and f is a Γ-filtration, then f is a zigzag morphism.

Chapter 13

The finite model property

13.1 Introduction

We have seen several examples of completeness results of the form

$$\vdash_{\mathsf{S}} \phi \quad \Leftrightarrow \quad \mathbb{M} \text{ models } \phi$$

where S is a standard system and $\mathbb{M}$ is a suitable class of structures (and ϕ is an arbitrary formula). However, in all of these cases there has been little or no information about the size of the structures in $\mathbb{M}$. In this chapter we investigate the consequences of the existence of such an equivalence for a class $\mathbb{M}$ of finite structures.

13.2 The fmp explained

Let S be any standard system and consider the following three classes of finite structures.

$\mathbb{F}$ = the class of finite valued structures which model S
$\mathbb{G}$ = the class of finite (unadorned) structures which model S
$\mathbb{H}$ = the class of separated valued structures which model S

For any of these classes $\mathbb{K}$, let $Th(\mathbb{K})$ be the set of all formulas ϕ modelled by $\mathbb{K}$. Clearly

$$Th(\mathbb{F}) \subseteq Th(\mathbb{H}) \quad , \quad Th(\mathbb{F}) \subseteq Th(\mathbb{G})$$

(since $\mathbb{H} \subseteq \mathbb{F}$ and each valuation on a member of $\mathbb{G}$ produces a member of $\mathbb{F}$).

Much of the usefulness of the fmp is due to the following result.

13.1 THEOREM. *The three sets*

$$Th(\mathbb{F}) \quad , \quad Th(\mathbb{G}) \quad , \quad Th(\mathbb{H})$$

are equal.

Proof. We show that

$$Th(\mathbb{G}) \subseteq Th(\mathbb{H}) \subseteq Th(\mathbb{F})$$

which, with the above observation, gives the required result.

For the first inclusion consider any $(\mathcal{A}, \alpha) \in \mathbb{H}$. From Exercise 12.1 we know that for each valuation μ on $\mathcal{A}$ and formula ϕ there is an appropriate substitution instance ϕ^μ of ϕ such that

$$(\mathcal{A}, \mu) \Vdash^v \phi \quad \Leftrightarrow \quad (\mathcal{A}, \alpha) \Vdash^v \phi^\mu.$$

In particular, since the axioms of S are closed under substitution and $(\mathcal{A}, \alpha)$ models S, we see that $(\mathcal{A}, \mu)$ also models S. Thus $\mathcal{A}$ models S and hence $\mathcal{A} \in \mathbb{G}$.

This shows that for each formula ϕ,

$$\phi \in Th(\mathbb{G}) \quad \Rightarrow \quad \mathcal{A} \Vdash^u \phi \quad \Rightarrow \quad (\mathcal{A}, \alpha) \Vdash^v \phi$$

and hence (since $(\mathcal{A}, \alpha)$ is an arbitrary member of $\mathbb{H}$) we have

$$\phi \in Th(\mathbb{G}) \quad \Rightarrow \quad \phi \in Th(\mathbb{H})$$

as required.

For the second inclusion consider any member $(\mathcal{A}, \alpha)$ of $\mathbb{F}$ and let $\approx$ be the semantic equivalence relation on A given by

$$a \approx b \Leftrightarrow \left\{ \begin{array}{c} \text{For all formulas } \phi, \\ a \Vdash \phi \quad \Leftrightarrow \quad b \Vdash \phi \end{array} \right\}$$

(for $a, b \in A$). We know we may slice $(\mathcal{A}, \alpha)$ by $\approx$ to produce a filtration

$$(\mathcal{A}, \alpha) \xrightarrow{f} (\mathcal{B}, \beta)$$

for which

$$(\mathcal{A}, \alpha, a) \Vdash \phi \quad \Leftrightarrow \quad (\mathcal{B}, \beta, f(a)) \Vdash \phi$$

for all formulas ϕ and $a \in A$. We know that $\mathcal{B}$ is separated, and hence $(\mathcal{B}, \beta) \in \mathbb{H}$. This is enough to give the required inclusion. ■

Now let

$$\mathsf{S}(fin) = Th(\mathbb{F}) = Th(\mathbb{G}) = Th(\mathbb{H})$$

so that

$$\vdash_{\mathsf{S}} \phi \quad \Rightarrow \quad \phi \in \mathsf{S}(fin)$$

(for all formulas ϕ).

13.2 DEFINITION. We say the system S has the *f(inite) m(odel) p(roperty)* if the above implication is, in fact, an equivalence. ■

Although it is not within the scope of this book, it is worth saying a few words about the consequences of the fmp concerning decidability.

Most formal system S that we are concerned with are such that the set $\mathsf{S}(all)$ of all formula ϕ for which

$$\vdash_{\mathsf{S}} \phi$$

is automatically recursive enumerable. This is the case when S is finitely axiomatizable. If such a system also has the fmp then, in fact, it is decidable, i.e. the set $\mathsf{S}(all)$ is recursive. To see this we use the equality $\mathsf{S}(all) = \mathsf{S}(fin)$ and the fact that the complement of $\mathsf{S}(fin)$ is recursively enumerable. To enumerate the complement of $\mathsf{S}(fin)$ first begin to enumerate all finite structures. Using the fact that S is finitely axiomatizable we can extract from this list an enumeration of all finite models of S. We then enumerate all the formulas which are not modelled by at least one structure in this second list. This list of formulas is the complement of $\mathsf{S}(fin)$.

We formally record this fact.

13.3 THEOREM. *Let* S *be a finitely axiomatizable system with the fmp. Then* S *is decidable.*

Let us now continue with our study of the fmp. First we observe that it is not necessary to use all finite models of a system.

13.4 LEMMA. *Suppose* S *is a standard system and* $\mathbb{M}$ *is any class of finite structures such that the equivalence*

$$\vdash_{\mathsf{S}} \phi \quad \Leftrightarrow \quad \mathbb{M} \textit{ models } \phi$$

holds. Then S *has the fmp.*

Proof. All the structures in $\mathbb{M}$ model S so that either

$$\mathbb{M} \subseteq \mathbb{F} \quad \text{or} \quad \mathbb{M} \subseteq \mathbb{G}$$

depending on whether the structures are valued or not. But then either

$$\mathsf{S}(fin) = Th(\mathbb{F}) \subseteq Th(\mathbb{M}) \quad \text{or} \quad \mathsf{S}(fin) = Th(\mathbb{G}) \subseteq Th(\mathbb{M})$$

so that for each formula ϕ

$$\phi \in \mathsf{S}(fin) \quad \Rightarrow \quad \phi \in Th(\mathbb{M}) \quad \Rightarrow \quad \vdash_{\mathsf{S}} \phi$$

which gives the required result. ■

The fmp is another way of obtaining completeness.

13.5 THEOREM. *Let* S *be a standard system with the fmp. Then* S *is Kripke-complete.*

Proof. Let $\mathbb{M}$ be the class of all the unadorned models of S. Then $\mathbb{G} \subseteq \mathbb{M}$ and for each formula ϕ

$$\mathbb{M} \text{ models } \phi \quad \Rightarrow \quad \mathbb{G} \text{ models } \phi \quad \Rightarrow \quad \vdash_{\mathsf{S}} \phi$$

where the second implication holds by the fmp. This gives the required Kripke-completeness. ■

Theorems 13.3 and 13.5 show that finitely axiomatizable systems with the fmp are rather pleasant. Thus it is worth looking at some examples of such systems.

13.3 The classic systems have the fmp

The classic systems are those monomodal systems whose axioms are the various combinations of the shapes

D, T, B, 4, 5.

(There are 15 such systems.) We use several different filtrations to show that each of these systems has the fmp.

To illustrate the method used let us begin with the minimal system K.

13.6 THEOREM. *The system* K *has the fmp.*

Proof. Consider any formula ϕ such that

$$\text{not}[\vdash_{\mathsf{S}} \phi].$$

It suffices to produce a finite valued structure which models K but which does not model this formula ϕ.

We know there is some valued structure $(\mathcal{A}, \alpha)$ which models K but does not model ϕ. This model may not be finite so we produce the required model by taking a slice of $(\mathcal{A}, \alpha)$.

Let Γ be the set of subformulas of ϕ and let

$$(\mathcal{A}, \alpha) \xrightarrow{f} (\mathcal{B}, \beta)$$

be any Γ-filtration (say the left-most one). Since Γ is finite, the structure $(\mathcal{B}, \beta)$ is finite and (trivially) models K. Also, for each $\gamma \in \Gamma$ and $a \in A$,

$$(\mathcal{A}, \alpha, a) \Vdash \gamma \quad \Leftrightarrow \quad (\mathcal{B}, \beta, f(a)) \Vdash \gamma.$$

Finally, there is some $a \in A$ with $a \Vdash \neg\phi$ and hence (since $\phi \in \Gamma$) we have $f(a) \Vdash \neg\phi$ so that $(\mathcal{B}, \beta)$ is the required model of K which does not model ϕ. ■

We now have to deal with the various axioms D, T, B, 4, and 5. For this we use a mixture of correspondence properties and preservation properties.

First a rather trivial result.

13.7 LEMMA. *Suppose*

$$\mathcal{A} \xrightarrow{f} \mathcal{B}$$

is a surjective morphism. Then both

(d) if $\mathcal{A}$ is serial then so is $\mathcal{B}$

(t) if $\mathcal{A}$ is reflexive then so is $\mathcal{B}$

hold.

Proof. (d) Consider any $b_1 \in B$. We must produce some $b_2 \in B$ with $b_1 \longrightarrow b_2$. Since f is surjective there is at least one $a_1 \in A$ with $f(a_1) = b_1$. Since $\mathcal{A}$ is serial there is some $a_2 \in A$ with $a_1 \to a_2$. Since f is a morphism this gives $f(a_1) \longrightarrow f(a_2)$ so we may set $b_2 = f(a_2)$.

(t) This is proved in a similar way. ■

This preservation result is sufficient to show that KD and KT have the fmp. However it is not strong enough to deal with the other axioms because other properties are not preserved by mere surjective morphisms. We need a tighter form of morphism, and this is where filtrations are useful. As an illustration let us see how we handle the shape B.

13.8 LEMMA. *Suppose*

$$(\mathcal{A}, \alpha) \xrightarrow{f} (\mathcal{B}, \beta)$$

is a left-most filtration where $\mathcal{A}$ is symmetric. Then $\mathcal{B}$ is also symmetric.

Proof. Consider any $b_1, b_2 \in B$ with $b_1 \longrightarrow b_2$. Since the filtration is left-most there are $a_1, a_2 \in A$ with

$$f(a_1) = b_1 \quad , \quad f(a_2) = b_2 \quad , \quad a_1 \longrightarrow a_2.$$

Since $\mathcal{A}$ is symmetric, we have $a_2 \longrightarrow a_1$ and hence, again since each filtration is a morphism, we have $b_2 \longrightarrow b_1$, as required. ■

We can now extend our class of examples of systems with the fmp.

13.9 THEOREM. *Let* S *be any of the 6 systems whose axioms are taken from*

$$D,\ T,\ B.$$

Then S *has the fmp.*

Proof. We know that there is a structural property $\| S \|$ such that for each structure $\mathcal{A}$

$$\mathcal{A} \text{ models } \mathsf{S} \quad \Leftrightarrow \quad \mathcal{A} \text{ has } \| S \| .$$

Also, from Lemmas 13.7 and 13.8, we know that for each left-most filtration

$$(\mathcal{A},\alpha) \xrightarrow{\ f\ } (\mathcal{B},\beta)$$

if $\mathcal{A}$ has $\| S \|$ then $\mathcal{B}$ has $\| S \|$.

Now consider any formula ϕ with

$$\text{not}[\vdash_{\mathsf{S}} \phi].$$

Since S is Kripke-complete, there is some structure $\mathcal{A}$ with property $\| S \|$ and which does not model ϕ, i.e. there is a valuation α on $\mathcal{A}$ such that $(\mathcal{A},\alpha)$ does not model ϕ.

Let Γ be the set of subformulas of ϕ and let f, as above, be the left-most Γ-filtration of $(\mathcal{A},\alpha)$. Note that $\mathcal{B}$ is is finite (since Γ is) and $\mathcal{B}$ has property $\| S \|$, and hence models S. Finally, since $\phi \in \Gamma$, we know, by the filtration property, that $(\mathcal{B},\beta)$ does not model ϕ, from which we obtain the required result. ■

We now turn to the shape 4 which, of course, characterizes transitivity. To deal with this we need a custom built filtration.

As usual, let Γ be any set of formulas which is closed under subformulas. Also let $(\mathcal{A},\alpha)$ be any structure, and consider the canonical Γ-quotient

$$A \xrightarrow{\ f\ } B$$

where

$$f(a_1) = f(a_2) \quad \Leftrightarrow \quad (\forall \gamma \in \Gamma)[a_1 \Vdash \gamma \Leftrightarrow a_2 \Vdash \gamma]$$

(for $a_1, a_2 \in A$). We convert B into a structure using the transition relation $\longrightarrow$ given by

$$b_1 \longrightarrow b_2 \ \Leftrightarrow \ \begin{cases} \text{For all } a_1 \in b_1, a_2 \in b_2 \text{ and} \\ \text{formulas } \phi \text{ with } \Box\phi \in \Gamma, \\ a_1 \Vdash \Box\phi \Rightarrow a_2 \Vdash \phi \wedge \Box\phi. \end{cases}$$

We impose on $\mathcal{B}$ the usual valuation β given by

$$b \in \beta(P) \quad \Leftrightarrow \quad (\exists a \in b)[a \in \alpha(P)]$$

(for $P \in \Gamma$ with other values of β unimportant).

This construction is sometimes known as the Lemmon filtration.

13.10 LEMMA. *Let $(\mathcal{A},\alpha)$ be any valued structure with $\mathcal{A}$ transitive. Then the above construction produces a Γ-filtration*

$$(\mathcal{A},\alpha) \xrightarrow{\ f\ } (\mathcal{B},\beta)$$

for which $\mathcal{B}$ is also transitive.

Proof. We verify the various conditions in turn.

To show that f is a morphism consider any $a_1, a_2 \in A$ with $a_1 \to a_2$. We must show that $f(a_1) \to f(a_2)$. To this end consider any $x_1 \in f(a_1)$ and $x_2 \in f(a_2)$. Then for each formula ϕ with $\Box\phi \in \Gamma$ we have

$$\begin{array}{rlll} x_1 \Vdash \Box\phi & \Rightarrow & a_1 \Vdash \Box\phi & \text{(since } f(x_1) = f(a_1)) \\ & \Rightarrow & a_1 \Vdash \Box\phi \wedge \Box^2\phi & \text{(since } \mathcal{A} \text{ is transitive)} \\ & \Rightarrow & a_2 \Vdash \phi \wedge \Box\phi & \text{(since } a_1 \to a_2) \\ & \Rightarrow & x_2 \Vdash \phi \wedge \Box\phi & \text{(since } f(a_2) = f(x_2)) \end{array}$$

as required.

The three conditions (Sur, Var, Fil) hold by construction.

Finally we must show that $\mathcal{B}$ is transitive. Thus, consider any $b_1, b_2, b_3 \in B$ with

$$b_1 \longrightarrow b_2 \longrightarrow b_3$$

and let $a_i \in b_i$ for $i = 1, 2, 3$. Then for each formula ϕ with $\Box\phi \in \Gamma$ we have

$$\begin{array}{rlll} a_1 \Vdash \Box\phi & \Rightarrow & a_2 \Vdash \phi \wedge \Box\phi & \\ & \Rightarrow & a_2 \Vdash \Box\phi & \Rightarrow \ a_3 \Vdash \phi \wedge \Box\phi \end{array}$$

so that $b_1 \longrightarrow b_3$ as required. ■

This result with Lemma 13.7 is enough to obtain the following result.

13.11 THEOREM. *The three systems*

$$\mathsf{K4} \ , \ \mathsf{KD4} \ , \ \mathsf{KT4} = \mathsf{S4}$$

all have the fmp.

Next we look at the combination of transitivity and symmetry, i.e. the models of KB4. Again we need a custom built filtration. Thus consider the usual data of a finite set of formulas Γ which is closed under subformulas, let $(\mathcal{A},\alpha)$ be any valued structure, and consider the canonical Γ-quotient

$$A \xrightarrow{\ f\ } B$$

We convert B into a structure $\mathcal{B}$ using a transition relation $\longrightarrow$ and then impose the usual valuation β on $\mathcal{B}$. For transitivity and symmetry the appropriate relation is defined as follows.

$$b_1 \longrightarrow b_2 \;\Leftrightarrow\; \begin{array}{l}\text{For all } a_1 \in b_1 \text{ and } a_2 \in b_2 \text{ and formulas } \phi \text{ with } \Box\phi \in \Gamma, \\ a_1 \Vdash \Box\phi \;\Rightarrow\; a_2 \Vdash \phi \wedge \Box\phi \\ \text{and} \\ a_2 \Vdash \Box\phi \;\Rightarrow\; a_1 \Vdash \phi \wedge \Box\phi\end{array}$$

I will leave the proof of the following to you.

13.12 LEMMA. *Let $(\mathcal{A}, \alpha)$ be any valued structure with $\mathcal{A}$ transitive and symmetric. Then the above construction produces a Γ-filtration*

$$(\mathcal{A}, \alpha) \xrightarrow{f} (\mathcal{B}, \beta)$$

for which $\mathcal{B}$ is also transitive and symmetric.

These results are enough to prove the following result.

13.13 THEOREM. *Let S be any of the 11 standard formal system whose axioms are made up of various combinations of the shapes D,T,B, and 4. Then S has the fmp.*

We now consider the systems formed by extending the above 11 systems by the addition of the axiom 5. This gives us four new systems K5, KD5 and K45, kD45.

The axiom 5 alone, which captures the euclidean property, is a little more delicate to deal with. We need a bit of a preamble.

Starting from any set Γ of formulas, which, as usual, we assume is finite and closed under subformulas, we first set

$$\Box\Gamma = \{\Box\phi \mid \phi \in \Gamma\} \quad , \quad \Diamond\Gamma = \{\Diamond\phi \mid \phi \in \Gamma\}$$

and then set

$$\Gamma^{\bullet} = \Gamma \cup \Box\Gamma \cup \Diamond\Gamma.$$

Thus $\Gamma \subseteq \Gamma^{\bullet}$ and $\Gamma^{\bullet}$ is also finite and close under subformulas. We now iterate this construction to get

$$\Gamma_0 = \Gamma \quad , \quad \Gamma_{r+1} = (\Gamma_r)^{\bullet}$$

for each $r < \omega$, and then set

$$\Gamma^{*} = \bigcup\{\Gamma_r \mid r < \omega\}.$$

In general Γ^* is closed under subformulas but need not be finite. However, when we work relative to the base system K5, we can retrieve finiteness.

By Exercise 8.6 of Chapter 8 we know that modulo K5 each formula has just finitely many modal variants. More precisely, we know that for each $\psi \in \Gamma^*$ there is some $\theta \in \Gamma^{\bullet\bullet}$ with

$$\vdash_{\mathsf{K5}} \psi \leftrightarrow \theta$$

so that Γ^* contains no more information than $\Gamma^{\bullet\bullet}$.

Consider any valued structure $(\mathcal{A}, \alpha)$ and let

$$A \xrightarrow{f} B$$

be the canonical $\Gamma^{\bullet\bullet}$-quotient. Let $\longrightarrow$ be the transition relation on B given by

$$b_1 \longrightarrow b_2 \;\Leftrightarrow\; \left\{\begin{array}{l}\text{For each } a_1 \in b_1, a_2 \in b_2 \text{ and formula } \phi \in \Gamma^{\bullet\bullet}, \\ a_1 \Vdash \Box\phi \Rightarrow a_2 \Vdash \phi \\ \text{and} \\ a_1 \Vdash \Diamond\phi \Leftarrow a_2 \Vdash \phi.\end{array}\right.$$

Note the direction of the second implication in this definition.

This gives us a structure $\mathcal{B}$ on which we impose the usual valuation.

13.14 THEOREM. *When the structure $\mathcal{A}$ is euclidean, the assignment f produces a $\Gamma^{\bullet\bullet}$-filtration*

$$(\mathcal{A}, \alpha) \xrightarrow{f} (\mathcal{B}, \beta)$$

where $\mathcal{B}$ is also euclidean.

Proof. We verify first that f is an unadorned morphism. Thus consider any $x_1, x_2 \in A$ with

$$x_1 \longrightarrow x_2$$

and any $a_1 \in f(x_1), a_2 \in f(x_2)$. For each $\phi \in \Gamma^{\bullet\bullet}$ there are $\psi, \theta \in \Gamma^{\bullet\bullet}$ with

$$\vdash_{\mathsf{K5}} \Box\phi \leftrightarrow \psi \quad , \quad \vdash_{\mathsf{K5}} \Diamond\phi \leftrightarrow \theta$$

and (since $\mathcal{A}$ models K5) both of these equivalences hold in $\mathcal{A}$. Hence

$$\begin{array}{rl} a_1 \Vdash \Box\phi & \Rightarrow a_1 \Vdash \psi \\ & \Rightarrow x_1 \Vdash \psi \\ & \Rightarrow x_1 \Vdash \Box\phi \\ & \Rightarrow x_2 \Vdash \phi \quad \Rightarrow a_2 \Vdash \phi \end{array}$$

where these implications follow by

- the above equivalence,
- the $\Gamma^{\bullet\bullet}$-induced equivalence,
- the above equivalence,
- the transition $x_1 \longrightarrow x_2$,
- the $\Gamma^{\bullet\bullet}$-induced equivalence.

A similar argument shows that

$$a_1 \Vdash \Diamond\phi \Leftarrow a_2 \Vdash \phi$$

and hence we get

$$f(x_1) \longrightarrow f(x_2)$$

to verify the morphism property.

The remaining properties required for a filtration are straight forward.

Finally, we must show that the target structure $\mathcal{B}$ is euclidean.

To this end consider any elements b_1, b_2, b_3 of $\mathcal{B}$ with

$$b_1 \nearrow b_2 , \quad b_1 \searrow b_3 .$$

We show that $b_2 \longrightarrow b_3$. Thus, consider any

$$a_1 \in b_1 \quad , \quad a_2 \in b_2 \quad , \quad a_3 \in b_3$$

and any formula $\phi \in \Gamma^{\bullet\bullet}$. Then

$$a_2 \Vdash \Box\phi \Rightarrow a_1 \Vdash \Diamond\Box\phi \Rightarrow a_1 \Vdash \Box\phi \Rightarrow a_3 \Vdash \phi$$

where these implications follow by

- the K5-equivalence trick used above and the definition of $b_1 \longrightarrow b_2$,
- the variant $\Diamond\Box\phi \to \Box\phi$ of the 5 axiom,
- the definition of $b_1 \longrightarrow b_3$.

A similar argument shows that

$$a_2 \Vdash \Diamond\phi \Leftarrow a_3 \Vdash \phi$$

and hence $b_2 \longrightarrow b_3$, as required. ■

A routine argument now gives the following.

13.15 THEOREM. *The two systems* K5 *and* KD5 *have the fmp.*

Finally, it remains to deal with K45 and KD45 which are the systems that capture transitive and euclidean (and serial) structures. Here we use the relation

$$b_1 \longrightarrow b_2 \quad \Leftrightarrow \quad \text{For all } a_1 \in b_1 \text{ and } a_2 \in a_2 \text{ and formulas } \phi \text{ with } \Box\phi \in \Gamma,$$
$$a_1 \Vdash \Box\phi \quad \Rightarrow \quad a_2 \Vdash \phi \wedge \Box\phi$$
and
$$a_1 \Vdash \Box\phi \quad \Leftrightarrow \quad a_2 \Vdash \Box\phi$$

and then impose the usual valuation on the structure.

13.16 LEMMA. *Let $(\mathcal{A}, \alpha)$ be any valued structure with $\mathcal{A}$ transitive and euclidean. Then the above construction produces a Γ-filtration*

$$(\mathcal{A}, \alpha) \xrightarrow{\ f\ } (\mathcal{B}, \beta)$$

for which $\mathcal{B}$ is also transitive and euclidean.

Proof. Assuming we have already shown that the construction does give a Γ-filtration, let us see why it preserves the euclidean property.

Thus consider any $b_1, b_2, b_3 \in B$ with

$$b_1 \nearrow b_2 , \quad b_1 \searrow b_3 .$$

We must show that $b_2 \longrightarrow b_3$. To this end consider any $a_1 \in b_1, a_2 \in b_2, a_3 \in b_3$ and consider any formula ϕ with $\Box\phi \in \Gamma$. Then the definition of $\longrightarrow$ on B gives

$$a_2 \Vdash \Box\phi \quad \Rightarrow \quad a_1 \Vdash \Box\phi \quad \Rightarrow \quad a_2 \Vdash \phi$$

and

$$a_2 \Vdash \Box\phi \Leftrightarrow a_1 \Vdash \Box\phi \Leftrightarrow a_3 \Vdash \Box\phi$$

so that $b_2 \to b_3$, as required. ■

Putting all of this together with various earlier results we have all the ingredients for the proof of the following.

13.17 THEOREM. *Let* S *be any standard system whose axioms are made up of various combinations of the shapes D, T, B, 4, and 5. Then* S

- *is finitely axiomatizable,*
- *is Kripke-complete,*
- *has the fmp,*
- *is decidable,*

and hence S *is very pleasant.*

I will leave you to provide the various details.

13.4 The basic temporal system has the fmp

So far we have used the filtration technique to obtain the fmp only for monomodal systems. However, the technique can also be used with polymodal systems, but some of the details are a little more intricate. As an example of this let us show that the basic temporal system TEMP has the fmp.

Recall that, from Chapter 8, Section 8.4 the models of this system are the temporal structures, i.e. the structures

$$\mathcal{A} = (A, \xrightarrow{+}, \xrightarrow{-})$$

where the two relations are converses of each other and both are transitive. Fix such a structure together with a valuation α on $\mathcal{A}$. Fix also some set Γ of formulas which is closed under subformulas. (In any application this will be the set of subformulas of some given formula.) The proof of the fmp boils down to the construction of a Γ-filtration

$$(\mathcal{A}, \alpha) \xrightarrow{f} (\mathcal{B}, \beta)$$

where $\mathcal{B}$ is also a temporal structure.

To do this consider the canonical Γ-quotient

$$\mathcal{A} \xrightarrow{f} B.$$

We will furnish B with two transition relations $\xrightarrow{+}, \xrightarrow{-}$ converting B into the required temporal structure $\mathcal{B}$, and then impose the usual valuation β on $\mathcal{B}$.

Thus define $\xrightarrow{+}, \xrightarrow{-}$ on B as follows.

$$\begin{array}{lll} b_1 \xrightarrow{+} b_2 & \Leftrightarrow & \text{For all } a_1 \in b_1, a_2 \in b_2 \text{ and for all formulas } \phi, \\ & & (+) \text{ if } \boxed{+}\phi \in \Gamma \text{ then} \\ & & \quad a_1 \Vdash \boxed{+}\phi \;\Rightarrow\; a_2 \Vdash \phi \wedge \boxed{+}\phi \\ & & (-) \text{ if } \boxed{-}\phi \in \Gamma \text{ then} \\ & & \quad a_2 \Vdash \boxed{-}\phi \;\Rightarrow\; a_1 \Vdash \phi \wedge \boxed{-}\phi \end{array}$$

$$\begin{array}{lll} b_1 \xrightarrow{-} b_2 & \Leftrightarrow & \text{For all } a_1 \in b_1, a_2 \in b_2 \text{ and for all formulas } \phi, \\ & & (-) \text{ if } \boxed{-}\phi \in \Gamma \text{ then} \\ & & \quad a_1 \Vdash \boxed{-}\phi \;\Rightarrow\; a_2 \Vdash \phi \wedge \boxed{-}\phi \\ & & (+) \text{ if } \boxed{+}\phi \in \Gamma \text{ then} \\ & & \quad a_2 \Vdash \boxed{+}\phi \;\Rightarrow\; a_1 \Vdash \phi \wedge \boxed{+}\phi \end{array}$$

Let

$$\mathcal{B} = (B, \xrightarrow{+}, \xrightarrow{-}).$$

We need to check several things.

13.18 LEMMA. *The function f is a morphism $\mathcal{A} \longrightarrow \mathcal{B}$.*

Proof. Consider first any $a_1, a_2 \in A$ with $a_1 \xrightarrow{+} a_2$. We must show that $f(a_1) \xrightarrow{+} f(a_2)$. To this end consider any $x_1 \in f(a_1), x_2 \in f(a_2)$ and any formula ϕ. Then:

if $\boxed{+}\phi \in \Gamma$ then

$$\begin{array}{llll} x_1 \Vdash \boxed{+}\phi & \Rightarrow & a_1 \Vdash \boxed{+}\phi & (\text{since } x_1 \simeq a_1) \\ & \Rightarrow & a_1 \Vdash \boxed{+}\phi \wedge \boxed{+}^2\phi & (\text{by axiom 4}) \\ & \Rightarrow & a_2 \Vdash \phi \wedge \boxed{+}\phi & (\text{since } a_1 \xrightarrow{+} a_2) \\ & \Rightarrow & x_2 \Vdash \phi \wedge \boxed{+}\phi & (\text{since } a_2 \simeq x_2) \end{array}$$

and

if $\boxed{-}\phi \in \Gamma$ then

$$\begin{array}{llll} x_2 \Vdash \boxed{-}\phi & \Rightarrow & a_2 \Vdash \boxed{-}\phi & (\text{since } x_2 \simeq a_2) \\ & \Rightarrow & a_2 \Vdash \boxed{-}\phi \wedge \boxed{-}^2\phi & (\text{by axiom 4}) \\ & \Rightarrow & a_1 \Vdash \phi \wedge \boxed{-}\phi & (\text{since } a_2 \xrightarrow{+} a_1) \\ & \Rightarrow & x_1 \Vdash \phi \wedge \boxed{-}\phi & (\text{since } a_1 \simeq x_1) \end{array}$$

which gives the first required result.

A similar argument also shows that

$$a_1 \xrightarrow{-} a_2 \quad \Rightarrow \quad f(a_1) \xrightarrow{-} f(a_2)$$

to complete the proof. ■

This with a couple of trivial observations shows that we have constructed a Γ-filtration

$$(\mathcal{A}, \alpha) \xrightarrow{f} (\mathcal{B}, \beta)$$

and so it remains to show that $\mathcal{B}$ is a temporal structure.

Clearly (by definition) the two relations $\xrightarrow{+}, \xrightarrow{-}$ on B are converses. Thus the final required piece of information is provided by the following.

13.19 LEMMA. *Both the relations of $\mathcal{B}$ are transitive.*

Proof. Consider any $b_1, b_2, b_3 \in B$ with

$$b_1 \xrightarrow{+} b_2 \xrightarrow{+} b_3$$

and consider any $a_1 \in b_1, a_2 \in b_2, a_3 \in b_3$. Then, for each formula ϕ,

$$\begin{array}{l} (+) \text{ if } \boxed{+}\phi \in \Gamma \text{ then} \\ \quad a_1 \Vdash \boxed{+}\phi \;\Rightarrow\; a_2 \Vdash \phi \wedge \boxed{+}\phi \\ \quad\quad\quad\quad\quad\;\, \Rightarrow\; a_2 \Vdash \boxed{+}\phi \;\Rightarrow\; a_3 \Vdash \phi \wedge \boxed{+}\phi \end{array}$$

and

$$(-) \text{ if } [-]\phi \in \Gamma \text{ then}$$
$$a_1 \Vdash [-]\phi \;\Rightarrow\; a_2 \Vdash \phi \wedge [-]\phi$$
$$\Rightarrow\; a_2 \Vdash [-]\phi \;\Rightarrow\; a_3 \Vdash \phi \wedge [-]\phi$$

which shows that $b_1 \xrightarrow{+} b_2$ and hence $\xrightarrow{+}$ is transitive.
A similar argument shows that $\xrightarrow{-}$ is also transitive. ■

I now expect you to fill in the details of the following result.

13.20 THEOREM. *The basic temporal system*

- *is finitely axiomatizable,*
- *is Kripke-complete,*
- *has the fmp,*
- *is decidable,*

and hence is very pleasant.

13.5 Exercises

13.1 Let S be a standard formal system axiomatized by a single sentence. By Exercise 10.1 of Chapter 10 we know that S is canonical. Now show that S has the fmp.

13.2 Show that a left-most filtration of a pathetic structure has a pathetic target. Hence show that

$$\mathsf{KP},\ \mathsf{KDP},\ \mathsf{KTP},\ \mathsf{KBP}$$

all have the fmp.

13.3 Recall that the shape

$$[k]\phi \rightarrow [i]\,[j]\phi$$

(for arbitrary ϕ) captures a certain composition property. Let $\mathcal{A}$ be the 4-element structure

$$\circ \xrightarrow{i} \circ \qquad \circ \xrightarrow{j} \circ$$

consisting of one i-transition, one j-transition, and no k-transitions. By considering a certain valuation on $\mathcal{A}$, show that the above formula need not be preserved by left-most filtrations.

13.4 Filtrations are useful because of their preservation properties.

(a) Show that for arbitrary labels i and j the shapes

(i)	$[i]\phi \rightarrow \langle j\rangle\phi$	(ii)	$[i]\phi \rightarrow [j]\phi$
(iii)	$\phi \rightarrow [j]\,\langle k\rangle\phi$	(iv)	$[i]\phi \rightarrow [j]\,\langle k\rangle\phi$

are preserved by left-most filtrations.

(b) Consider that particular case of (ii) where i labels $\Box$ and j labels $\Box^2$. Thus the shape $\Box\phi \rightarrow \Box^2\phi$, i.e. transitivity, is preserved by left-most filtrations. What, if anything, is wrong with this argument?

13.5 Consider the formal system KE of Exercise 10.2 of Chapter 10.

(a) Show that KE has the fmp.

(b) Show that the extensions KBE and KDE have the fmp.

(c) Using a modified Lemmon filtration, show that KE4 has the fmp.

13.6 Consider the formal system KF of Exercise 10.2 of Chapter 10.

(a) Show that KF, KBF and KDF have the fmp.

(b) Show that KF4 has the fmp.

13.7 Consider the two shapes E and F of Exercise 10.2 of Chapter 10.

(a) Show that KE5 has the fmp.

(b) What can you say about KF5?

13.8 Recall the notion of a *good* structure introduced in Exercise 4.12, namely, a
monomodal structure which is transitive and serial and such that each of its elements can see a reflexive element. Suppose the monomodal valued structure $(\mathcal{A},\alpha)$ is transitive and serial (but not necessarily good). For an arbitrary formula ϕ let Γ be the set of subformulas of ϕ and let

$$(\mathcal{A},\alpha) \longrightarrow (\mathcal{B},\beta)$$

be the Lemmon Γ-filtration.

(a) Show that $\mathcal{B}$ is good.

(b) Show there is no formula whose models are precisely the class of good structures.