Chapter 4

Valuation and satisfaction

4.1 Introduction

For each signature I we have now defined two quite different entities; the polymodal language of signature I, and the class of structures (labelled transition structures) of signature I. These two entities must now be made to interact. Thus the structures will be made to support a semantics for the language, or, equivalently, the language will be used to describe properties of structures.

The polymodal language has the usual propositional facilities together with a family of new 1-ary connectives $[i]$, one for each label i. We now wish to evaluate each formula of this language, i.e. determine whether or not a formula ϕ is TRUE or FALSE. Of course, this can not be done in isolation, we need to work in an appropriate context. To determine the truth value of ϕ we need three pieces of information together with an agreed procedure for using the information.

1. We need to know the truth values of the variables appearing in ϕ. As in the propositional case this information will be conveyed by a *valuation*, however, these modal valuations are more complicated than the propositional versions.

2. We need to know how to handle the propositional connectives. This will be done in exactly the same way as the propositional language (i.e. using the defining truth tables of the connectives). In this sense, modal logic subsumes propositional logic.

3. We need to know how to handle the modal connectives $[i]$. This will be done by working relative to a given structure $\mathcal{A}$. The relation $\xrightarrow{i}$ of this structure will control the connective $[i]$. The precise way this is done will be described in due course.

Thus, we are going to define a relation

$$(\mathcal{A}, \alpha, a) \Vdash \phi \qquad (4.1)$$

between

structures $\mathcal{A}$, valuations α , elements a of $\mathcal{A}$, and formulas ϕ.

This relation may be read as:

> Under the circumstances determined by $(\mathcal{A}, \alpha, a)$, the formula ϕ is forced to be true.

As can be expected, this relation (4.1) is defined by recursion on the complexity of ϕ. In this recursion, the two parameters $\mathcal{A}$ and α are held fixed throughout, but the parameter a must be allowed to vary through all elements of $\mathcal{A}$.

In the majority of places where we use this satisfaction relation, we may suppress the parameters $\mathcal{A}$ and α and abbreviate (4.1) to

$$a \Vdash \phi.$$

We may also read this as

$$a \quad \text{forces} \quad \phi$$

this saves quite a bit of space (and mental breath) and allows us to concentrate on the important point, namely how the element a regards the truth status of ϕ. In circumstances where it seems helpful or will avoid misunderstandings, we will use the expanded form (4.1).

What is the appropriate notion of a model valuation of α? The information that α must provide is, for each variable P, at which elements of the supporting structure $\mathcal{A}$ is P regarded as TRUE, and at which elements is P regarded as FALSE. This information is supplied in the following fashion.

4.1 DEFINITION. A *valuation* α on a structure is an assignment

$$\alpha : Var \longrightarrow \mathcal{P}A$$

from variables P to subsets $\alpha(P)$ of A. The pair $(\mathcal{A}, \alpha)$ is then a *valued structure.* ■

The idea here is that the variable P is regarded as TRUE at the element a precisely when $a \in \alpha(P)$. This provides the base case in the definition of (4.1), namely

$$a \Vdash P \quad \Leftrightarrow \quad a \in \alpha(P).$$

The recursion steps across the propositional connectives are the obvious ones, so all that remains (to complete the definition of (4.1)) is to describe how to handle

$$a \Vdash [i]\phi$$

for a label i and formula ϕ. Before we do this (in the next section) it is worthwhile looking at the nature of the structures involved.

For each signature I we have introduced three different *kinds* of associated structures. Firstly, we have the unadorned structures $\mathcal{A}$, i.e. the labelled transition structures described in Chapter 3. Each such structure can be enriched by a valuation α to form a *valued structure* $(\mathcal{A}, \alpha)$; and then we may distinguish a particular element a to form a *pointed valued structure* $(\mathcal{A}, \alpha, a)$. It is important not to confuse these three different kinds. We can not say that one kind is more important than the others; all three kinds have a role to play.

4.2 The basic satisfaction relation

So how do we handle the passage across a box $[i]$? To determine whether or not

$$a \Vdash [i]\phi$$

holds we must survey all the elements x with $a \xrightarrow{i} x$ and for each such x determine whether or not

$$x \Vdash \phi$$

holds. Thus, the full and precise definition of $\Vdash$ is as follows.

4.2 DEFINITION. Let $(\mathcal{A}, \alpha)$ be a given valued structure. The relation

$$a \Vdash \phi$$

between elements a of $\mathcal{A}$ and formulas ϕ is defined by recursion on ϕ (with variations of the parameter a) using the following clauses.

(Const) For the constants

$$a \Vdash \top \quad , \quad \text{not}[a \Vdash \bot].$$

(Var) For each variable P

$$a \Vdash P \Leftrightarrow a \in \alpha(P).$$

($\neg$) For each formula ϕ

$$a \Vdash \neg\phi \Leftrightarrow \text{not}[a \Vdash \phi].$$

($\wedge, \vee, \rightarrow$) For all formulas θ, ψ

$$\begin{aligned} a \Vdash (\theta \wedge \psi) &\Leftrightarrow a \Vdash \theta \text{ and } a \Vdash \psi \\ a \Vdash (\theta \vee \psi) &\Leftrightarrow a \Vdash \theta \text{ or } a \Vdash \psi \\ a \Vdash (\theta \rightarrow \psi) &\Leftrightarrow a \Vdash \psi \text{ whenever } a \Vdash \theta. \end{aligned}$$

(i) For each label i and formula ϕ

$$a \Vdash [i]\phi \Leftrightarrow (\forall x \prec_i a)[x \Vdash \phi]$$

where the quantified variable x ranges over A (the carrying set of $\mathcal{A}$). ■

The right hand side of the equivalence in clause (i) is an abbreviated version of

$$(\forall x)[x \prec_i a \Rightarrow x \Vdash \phi].$$

The condensed version will prove to be very convenient in many computations.

In Section 4.1 we gave the formal reading of (4.1) together with a suggested shortening 'a forces ϕ'. For variety we will also read (4.1) as

$$\begin{array}{rcl} \phi & \text{is valid at} & a, \\ \phi & \text{holds at} & a, \\ (\mathcal{A}, \alpha, a) & \text{satisfies} & \phi, \\ (\mathcal{A}, \alpha, a) & \text{models} & \phi, \end{array}$$

or in various other similar ways.

Observe that each pointed valued structure $(\mathcal{A}, \alpha, a)$ gives us a 2-valuation ν_a where, for a variable P,

$$\nu_a(P) = \begin{cases} \text{TRUE} & \text{if } a \in \alpha(P) \\ \text{FALSE} & \text{if } a \notin \alpha(P). \end{cases}$$

We then see that for each propositional (i.e. box-free) formula ϕ,

$$a \Vdash \phi \quad \Leftrightarrow \quad [\![\phi]\!]_{\nu_a} = \text{TRUE}$$

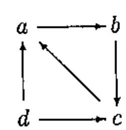
where, of course, $[\![\cdot]\!]_{\nu_a}$ is the assignment induced by the 2-valuation ν_a as constructed in Chapter 1. In particular we obtain the following.

4.3 PROPOSITION. *If ϕ is a propositional tautology then ϕ is satisfied by every pointed valued structure.*

4.3 Some examples

The relation $\Vdash$ is probably the single most important notion in the whole of this book (and, in fact, the whole of modal logic). It is therefore worth spending some time looking at particular examples to help us develop a feel for the relation.

Consider the 4-element monomodal structure



where no element is reflexive. Consider also any valuation α with

$$\alpha(P) = \{a, c\}$$

for some variable P. Thus

$$a \Vdash P \quad , \quad b \Vdash \neg P \quad , \quad c \Vdash P \quad , \quad d \Vdash \neg P.$$

We also see that

$$a \Vdash \Box\neg P \quad , \quad b \Vdash \Box P \quad , \quad c \Vdash \Box P$$

for the only element x with, respectively

$$a \longrightarrow x \quad , \quad b \longrightarrow x \quad , \quad c \longrightarrow x$$

is

$$x = b \quad , \quad x = c \quad , \quad x = a.$$

Next we see that

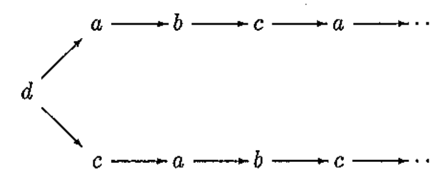
$$a \Vdash \Box^2 P \quad , \quad b \Vdash \Box^2 P \quad , \quad c \Vdash \Box^2 \neg P.$$

For instance, to verify the first we must show that for all pairs x, y with

$$a \longrightarrow x \longrightarrow y$$

we have $y \Vdash P$. But the only such pair is $x = b$ and $y = c$, so we are done. A similar argument verifies the other two cases.

The element d is slightly more interesting. If we develop all the paths starting from d then we get



from which we find that

$$d \Vdash \Box P \quad , \quad d \Vdash \neg\Box^2 P \quad , \quad d \Vdash \neg\Box^3 P \quad , \quad d \Vdash \Box^4 P$$

etc.

Notice that the only path of length 3 starting from a is

$$a \longrightarrow b \longrightarrow c \longrightarrow a.$$

Thus, for any formula ϕ,

$$\begin{array}{rcl} a \Vdash \Box\phi & \Leftrightarrow & b \Vdash \phi \\ a \Vdash \Box^2\phi & \Leftrightarrow & c \Vdash \phi \\ a \Vdash \Box^3\phi & \Leftrightarrow & a \Vdash \phi \end{array}$$

so that
$$a \Vdash (\Box^3\phi \leftrightarrow \phi).$$
For similar reasons we see that $\Box^3\phi \leftrightarrow \phi$ also holds at b and c. Furthermore this argument is valid no matter which valuation is involved.

A similar, but slightly more complicated argument shows that
$$d \Vdash (\Box^4\phi \leftrightarrow \Box\phi)$$
and again this is independent of the valuation involved. This shows that the formula $\Box^4\phi \leftrightarrow \Box\phi$ holds at every element of $\mathcal{A}$ no matter which valuation is carried. In the notation introduced later (in the next section), this shows that
$$\mathcal{A} \Vdash^u \Box^4\phi \leftrightarrow \Box\phi.$$

As explained in Chapter 2, the diamond $\langle i \rangle$ may be introduced into the polymodal language as an abbreviation in the sense that, for each formula ϕ
$$\langle i \rangle\phi \quad \text{abbreviates} \quad \neg[i]\neg\phi.$$
It is instructive to see how the forcing relation $\Vdash$ handles this. The result shouldn't be a surprise.

4.4 PROPOSITION. *Let $(\mathcal{A}, \alpha)$ be a valued structure. Then the equivalence*
$$a \Vdash \langle i \rangle\phi \quad \Leftrightarrow \quad (\exists x \prec_i a)[x \Vdash \phi]$$
holds for all labels i, elements a, and formulas ϕ.

Proof. We use the standard manipulation of quantifiers. Using '$\neg$' for both the formal and the informal negation we have, applying the appropriate clauses of Definition 4.2
$$\begin{aligned} a \Vdash \langle i \rangle\phi &\Leftrightarrow a \Vdash \neg[i]\neg\phi \\ &\Leftrightarrow \neg[a \Vdash [i]\neg\phi] \\ &\Leftrightarrow \neg(\forall x \prec_i a)[x \Vdash \neg\phi] \\ &\Leftrightarrow (\exists x \prec_i a)\neg[x \Vdash \neg\phi] \quad \Leftrightarrow \quad (\exists x \prec_i a)[x \Vdash \phi] \end{aligned}$$
as required. ■

It is worth comparing the equivalence of Proposition 4.4 with clause (i) of Definition 4.2. There are many similarities between the box connective $\Box$ and the universal quantifier $\forall$, and between the diamond connective $\Diamond$ and the existential quantifier $\exists$. Once this has been grasped, many of the computations of modal logic become almost routine. For instance, it is easy to check that
$$\begin{aligned} a &\Vdash [i][j]\phi &\Leftrightarrow\ & (\forall x \prec_i a)(\forall y \prec_j x)[y \Vdash \phi] \\ a &\Vdash [i]\langle j \rangle\phi &\Leftrightarrow\ & (\forall x \prec_i a)(\exists y \prec_j x)[y \Vdash \phi] \\ a &\Vdash \langle i \rangle[j]\phi &\Leftrightarrow\ & (\exists x \prec_i a)(\forall y \prec_j x)[y \Vdash \phi] \\ a &\Vdash \langle i \rangle\langle j \rangle\phi &\Leftrightarrow\ & (\exists x \prec_i a)(\exists y \prec_j x)[y \Vdash \phi]. \end{aligned}$$

There are several particular valuations which ensure that some simple formulas are valid. For instance, for a given element a of the structure $\mathcal{A}$ and for a given variable P, consider any valuation α such that $\alpha(P) = \{a\}$. Then, for each $x \in A$,
$$x \Vdash P \quad \Leftrightarrow \quad x = a$$
and, trivially,
$$a \Vdash P.$$
Similarly, with a valuation α such that
$$\alpha(P) = \{x \in A \mid x \prec a\}$$
we have, for $x \in A$,
$$x \Vdash P \quad \Leftrightarrow \quad x \prec a$$
and
$$a \Vdash \Box P.$$
In the same way, with a valuation such that
$$x \Vdash P \quad \Leftrightarrow \quad (\exists y)[x \prec y \prec a]$$
we have
$$a \Vdash \Box^2 P.$$

A similar, but slightly more complicated, example deserves a little more formality. In this example we use the $*$-closure (i.e. reflexive, transitive closure) $\prec*$ of a relation $\prec$. If you are not familiar with this notion, a full discussion is given later in Chapter 14, Section 14.2.

4.5 PROPOSITION. *For a given element a of a structure $\mathcal{A}$ and a given variable P, consider any valuation on $\mathcal{A}$ such that*
$$x \Vdash P \quad \Leftrightarrow \quad (\forall y)[y \prec* x \Rightarrow y \prec a]$$
(for $x \in A$). Then
$$a \Vdash \Box(\Box P \to P)$$
holds.

Proof. Consider $b \prec a$ with
$$b \Vdash \Box P.$$
We require that $b \Vdash P$. To this end consider any $y \prec* b$. Then, either
$$y = b \quad \text{or} \quad (\exists x)[y \prec* x \prec b]$$
so that either
$$y \prec a \quad \text{or} \quad (\exists x)[y \prec* x \Vdash P]$$
and hence, in both cases, $y \prec a$. Thus $b \Vdash P$, as required. ■

4.4 The three satisfaction relations

So far we have defined only the basic satisfaction relation

$$(\mathcal{A}, \alpha, a) \Vdash \phi$$

for a pointed valued structure. We now allow this to ramify into three related satisfaction relations

$$\Vdash^p \quad , \quad \Vdash^v \quad , \quad \Vdash^u$$

all of which are useful at one place or another. It is extremely important that you learn to distinguish between these three satisfaction relations; the second and third are both derived from the first, but none can be said to be more fundamental than the other two.

The relation $\Vdash^p$ is just the relation $\Vdash$, and so it is used with pointed valued structures (hence the decoration p).

The relation $\Vdash^v$ is used with valued structures and is formed from $\Vdash^p$ by quantifying out the point. Thus, by definition,

$$(\mathcal{A}, \alpha) \Vdash^v \phi \quad \Leftrightarrow \quad (\forall a)[(\mathcal{A}, \alpha, a) \Vdash^p \phi]$$

where the quantified variable a ranges over all elements of $\mathcal{A}$.

Finally the relation $\Vdash^u$ is used with unadorned structures and is formed from $\Vdash^v$ by quantifying out the valuation. Thus

$$\mathcal{A} \Vdash^u \phi \quad \Leftrightarrow \quad (\forall \alpha)[(\mathcal{A}, \alpha) \Vdash^v \phi]$$

where the quantified variable α ranges over all valuations on $\mathcal{A}$.

All three give explications of the modelling process. Thus when we speak of 'a model' of a formula ϕ we could mean an unadorned structure, a valued structure, or a pointed valued structure. In all cases we should make sure we understand exactly which notion is intended.

Let us look at some examples of the differing properties of $\Vdash^p$, $\Vdash^v$, and $\Vdash^u$.

Consider the 4-element example $(\mathcal{A}, \alpha)$ described at the beginning of Section 4.3. There we have

$$(\mathcal{A}, \alpha, a) \Vdash^p P \quad \text{but} \quad \text{not}[(\mathcal{A}, \alpha) \Vdash^v P]$$

(for consider the element b). Notice also that

$$\text{not}[(\mathcal{A}, \alpha) \Vdash^v \neg P]$$

so that negations can not be transferred across $\Vdash^v$ (as they can with $\Vdash^p$). Next observe that

$$(\mathcal{A}, \alpha) \Vdash^v P \to \Box^3 P$$

for the only elements x with $x \Vdash P$ are $x = a$ and $x = c$, and each path of length three from one of these elements returns to its starting element. However, it is easy to construct a different valuation β or $\mathcal{A}$ for which

$$\text{not}[(\mathcal{A}, \beta) \Vdash^v \neg P \to \Box^3 P]$$

(for instance, let $\beta(P) = \{d\}$.) Thus

$$\text{not}[\mathcal{A} \Vdash^u \neg P \to \Box^3 P].$$

Finally, as we noted before, we have

$$\mathcal{A} \Vdash^u \Box^4 P \leftrightarrow \Box P.$$

Sometimes simple properties of a structure $\mathcal{A}$ ensure that it models certain formulas. For example, consider the four properties reflexivity, transitivity, being pathetic, and denseness of a relation used in Chapter 3, Section 3.4. The following result should be compared to Proposition 3.4.

4.6 LEMMA. *For a given structure $\mathcal{A}$, suppose a distinguished relation $\prec$ is, respectively,*

(i) Reflexive
(ii) Transitive
(iii) Pathetic
(iv) Dense.

Then, in each case, for each formula ϕ, the corresponding compound formula

(i) $\Box\phi \to \phi$
(ii) $\Box\phi \to \Box^2\phi$
(iii) $\phi \to \Box\phi$
(iv) $\Box^2\phi \to \Box\phi$

is modelled by $\mathcal{A}$.

Proof. For instance, suppose that $\prec$ is transitive and, for an arbitrary valuation on $\mathcal{A}$ and element a, suppose that

$$a \Vdash \Box\phi.$$

To show that $a \Vdash \Box^2\phi$, consider elements $c \prec b \prec a$. Then, since $\prec$ is transitive, we have $c \prec a$, so that $c \Vdash \phi$, which is enough to verify case (ii).

The other three cases are verified in a similar way. ■

There are two properties of the satisfaction relations which are particularly important. These will be used later to form the basis of a proof system for modal logic. The first property provides the basic axioms.

4.7 LEMMA. *For each structure $\mathcal{A}$*

$$\mathcal{A} \Vdash^u [i](\theta \to \psi) \to ([i]\theta \to [i]\phi)$$

holds for all labels i and formulas θ and ψ.

Proof. Consider any valuation on $\mathcal{A}$ and element a of $\mathcal{A}$ such that

$$a \Vdash [i](\theta \to \psi) \quad , \quad a \Vdash [i]\theta.$$

Then, for each element $x \prec_i a$, we have

$$x \Vdash \theta \to \psi \quad , \quad x \Vdash \theta$$

so that $x \Vdash \psi$, and hence $a \Vdash [i]\psi$, which gives the required result. ■

The final result of this section will eventually provide the rules of inference for modal proof systems.

4.8 LEMMA. *For each valued structure $(\mathcal{A}, \alpha)$ the implication*

$$(\mathcal{A}, \alpha) \Vdash^v \phi \Rightarrow (\mathcal{A}, \alpha) \Vdash^v [i]\phi$$

holds for all labels i and formulas ϕ.

Proof. Suppose that $(\mathcal{A}, \alpha) \Vdash^v \phi$ and consider any pair of elements $x \prec_i a$. Then $x \Vdash \phi$ so that $a \Vdash [i]\phi$, and hence $(\mathcal{A}, \alpha) \Vdash^v [i]\phi$, as required. ■

You may be tempted to think that Lemma 4.8 could be improved to

$$(\mathcal{A}, \alpha) \Vdash \phi \to [i]\phi$$

or perhaps to

$$a \Vdash \phi \Rightarrow a \Vdash [i]\phi.$$

Neither of these hold in general, and you should look for appropriate counter-examples.

4.5 Semantics for modal algebras

As explained in Chapter 3, each structure $\mathcal{A}$ is equivalent to a modal algebra based on $\mathcal{P}A$. Any semantics supported by $\mathcal{A}$ can be transferred to this modal algebra, as we now describe.

By definition, a valuation on $\mathcal{A}$ is a mapping

$$\alpha : Var \longrightarrow \mathcal{P}A.$$

The forcing relation $\Vdash$ induced by α extends α to a mapping

$$Form \longrightarrow \mathcal{P}A$$

where, for each formula ϕ, the assigned subset of A is written

$$[\![\phi]\!]_\alpha.$$

This set is a measure of how true ϕ is in $(\mathcal{A}, \alpha)$.

In the definition of $[\![\phi]\!]_\alpha$ we use the boolean operations

$$\cap \,,\, \cup \,,\, \neg$$

of union, intersection, and complementation on $\mathcal{P}A$, together with the modal operations induced by the relations $\prec_i$. It is worth comparing this definition with Definitions 1.2, 2.3, and 4.2.

4.9 DEFINITION. Let $(\mathcal{A}, \alpha)$ be a given valued structure. For each formula ϕ the subset

$$[\![\phi]\!]_\alpha$$

of A is defined by recursion on ϕ using the following clauses.

(Const) For the constants

$$[\![\top]\!]_\alpha = A \quad , \quad [\![\bot]\!]_\alpha = \emptyset.$$

(Var) For each variable P

$$[\![P]\!]_\alpha = \alpha(P).$$

($\neg$) For each formula ϕ

$$[\![\neg\phi]\!]_\alpha = \neg[\![\phi]\!]_\alpha.$$

($\wedge, \vee, \to$) For all formulas θ, ψ

$$\begin{aligned} [\![\theta \wedge \psi]\!]_\alpha &= [\![\theta]\!]_\alpha \cap [\![\psi]\!]_\alpha \\ [\![\theta \vee \psi]\!]_\alpha &= [\![\theta]\!]_\alpha \cup [\![\psi]\!]_\alpha \\ [\![\theta \to \psi]\!]_\alpha &= \neg[\![\theta]\!]_\alpha \cup [\![\psi]\!]_\alpha \end{aligned}$$

(i) For each label i and formula ϕ

$$[\![\,[i]\phi]\!]_\alpha = [i]([\![\phi]\!]_\alpha)$$

where $[i]$ is the modal operation on $\mathcal{P}A$ induced by $\prec_i$. ■

It is clear from this definition that for the most part the distinguishing subscript on

$$[\![\phi]\!]_\alpha$$

is playing no useful purpose. We will therefore omit it in future unless this could lead to confusion (e.g. when there are two valuations around).

The two notions $\Vdash$ and $[\![\cdot]\!]$ are connected in an obvious way. The proof of this is entirely routine and will be left as an exercise.

4.10 PROPOSITION. *For each valued structure $(\mathcal{A}, \alpha)$ the equivalence*

$$a \Vdash \phi \quad \Leftrightarrow \quad a \in [\![\phi]\!]$$

holds for all elements a of $\mathcal{A}$ and formula ϕ.

Note how this shows that

$$(\mathcal{A}, \alpha) \Vdash^v \phi \quad \Leftrightarrow \quad [\![\phi]\!] = A$$

which can be useful in certain situations.

This rephrasing of the semantics provides a convenient way of evaluating substitution instances.

Recall that a substitution is a map

$$\sigma : Var \longrightarrow Form$$

which extends to a map

$$\begin{aligned} Form &\longrightarrow Form \\ \phi &\longmapsto \phi^\sigma. \end{aligned}$$

These should be compared with the maps α and $\phi \mapsto [\![\phi]\!]_\alpha$; the only difference between the two cases is that $Form$ is the target set for substitutions whereas $\mathcal{P}A$ is the target set for valuations.

The question to consider is how can we determine the value

$$[\![\phi^\sigma]\!]_\alpha$$

without first performing the substitution and then applying $[\![\cdot]\!]_\alpha$. The method is reminiscent of the way we dealt with iterated substitutions.

4.11 DEFINITION. For each substitution σ and valuation α, let $\alpha * \sigma$ be the valuation given by

$$(\alpha * \sigma)(P) \quad = \quad [\![P^\sigma]\!]_\alpha$$

for each variable P. ■

The following result is the analogue of Exercise 2.5(a) of Chapter 2.

4.12 THEOREM. *For each substitution σ, valuation α (on a structure), and formula ϕ we have*

$$[\![\phi^\sigma]\!]_\alpha \quad = \quad [\![\phi]\!]_\beta$$

where $\beta = \alpha * \sigma$.

Proof. This follows by recursion on ϕ. The various steps are entirely routine. For instance, for the step across a box we know that

$$([i]\phi)^\sigma \quad = \quad [i]\phi^\sigma$$

so that

$$[\![([i]\phi)^\sigma]\!]_\alpha = [\![[i]\phi^\sigma]\!]_\alpha = [i]([\![\phi^\sigma]\!]_\alpha) = [i][\![\phi]\!]_\beta = [\![[i]\phi]\!]_\beta$$

where the third equality follows by the induction hypothesis and the others by Definition 4.9 (for both α and β). ■

This result has a consequence for the satisfaction relation $\Vdash$ which, at times, can be very useful.

4.13 PROPOSITION. *Suppose ψ is a substitution instance of the formula ϕ. Then the implication*

$$\mathcal{A} \Vdash^u \phi \quad \Rightarrow \quad \mathcal{A} \Vdash^u \psi$$

holds for all structures $\mathcal{A}$.

Proof. Let σ be a substitution such that $\psi = \phi^\sigma$, and suppose $\mathcal{A} \Vdash \phi$. Consider any valuation α on $\mathcal{A}$. We require $[\![\psi]\!]_\alpha = A$. Let $\beta = \alpha * \sigma$. Then

$$[\![\psi]\!]_\alpha = [\![\phi^\sigma]\!]_\alpha = [\![\phi]\!]_\beta = A$$

where the last equality holds since $\mathcal{A} \Vdash^u \phi$. ■

As a simple application of this result we may combine it with Proposition 4.3 to obtain the following.

4.14 COROLLARY. *Suppose ψ is an instance of a propositional tautology. Then ψ holds in all structures.*

4.6 Exercises

4.1 Consider the 10 essentially different transition structures on two elements listed in Exercise 3.1 of Chapter 3. You are invited to determine which of the standard shapes D,T,B,4,... are modelled by these various structures. The table below gives a partial list of this information. You should check these result and fill in the other details.

	D	T	B	4	5	P	Q	R	G	L	M
(1)	×	×	.	✓	✓	.	✓	✓	✓	.	×
(2)	.	.	✓	✓	.	✓	.	✓	✓	×	.
(4)	.	✓	✓	.	✓	✓	✓	.	.	×	✓
(5)	×	×	.	✓	×	×	.	.	×	.	×
(7)	×	.	×	✓	.	.	×	.	×	×	×
(8)	.	×	×	.	✓	×	.	✓	×	×	.
(11)	✓	✓	.	✓	✓	×	×	✓	.	×	✓
(13)	✓	.	✓	.	.	×	✓	.	×	.	.
(14)	.	×	✓	×	×	.	×	.	.	×	×
(16)	✓	✓	✓	✓	.	×	×	.	×	×	×

4.2 Consider the three element monomodal transition structures of Exercise 3.2. Show that there are at least two non-isomorphic such structures which model none of the shapes D,T,...,M.

4.3 We may use the various order relations on N to define four different transition structures

$$\mathcal{N} = (\mathsf{N}, \longrightarrow)$$

where

$$\begin{array}{llll} (a) & x \longrightarrow y & \Leftrightarrow & x < y \\ (b) & x \longrightarrow y & \Leftrightarrow & x \leq y \\ (c) & x \longrightarrow y & \Leftrightarrow & x > y \\ (d) & x \longrightarrow y & \Leftrightarrow & x \geq y \end{array}$$

for $x, y \in \mathsf{N}$. The following table indicates whether or not some of the standard formulas hold in $\mathcal{N}$. You are invited to check this information and fill in the rest of the table.

	D	T	B	4	5	P	Q	R	G	L	M
(a)	✓	.	×	✓	×	×	×	.	✓	×	.
(b)	.	✓	×	✓	×	.	×	✓	✓	.	×
(c)	×	×	×	.	×	×	×	.	×	.	.
(d)	✓	✓	.	✓	.	×	.	✓	.	×	✓

4.4 Each of the sets

$$A := \mathsf{N}, \mathsf{Z}, \mathsf{Q}, \mathsf{R}$$

gives us a monomodal structure $\mathcal{A} = (A, \longrightarrow)$ where

$$a \longrightarrow b \Leftrightarrow a < b$$

for each $a, b \in A$. Let us write $\mathcal{N}, \mathcal{Z}, \mathcal{Q}, \mathcal{R}$ for these structures.

For each formula ϕ consider the compound formulas

$$\begin{array}{lll} \mathrm{T}(\phi) & := & \Box\phi \to \phi \\ \mathrm{L}(\phi) & := & \Box\mathrm{T}(\phi) \to \Box\phi \\ \mathrm{S}(\phi) & := & \Diamond\Box\phi \to \mathrm{L}(\phi) \end{array} \qquad \begin{array}{lll} \mathrm{U}(\phi) & := & \Box(\phi \to \Box\phi) \to \phi \\ \mathrm{V}(\phi) & := & \Box\mathrm{U}(\phi) \to \Box\phi \\ \mathrm{W}(\phi) & := & \Diamond\Box\phi \to \mathrm{V}(\phi). \end{array}$$

Show the following

(i)	$\mathcal{N} \Vdash^p \mathrm{S}(\phi)$	(ii)	$\mathcal{N} \Vdash^p \mathrm{W}(\phi)$
(iii)	$\mathcal{Z} \Vdash^p \mathrm{S}(\phi)$	(iv)	$\mathcal{Z} \Vdash^p \mathrm{W}(\phi)$
(v)	$\neg[\mathcal{Q} \Vdash^p \mathrm{S}(P)]$	(vi)	$\neg[\mathcal{Q} \Vdash^p \mathrm{W}(\phi)]$
(vii)	$\neg[\mathcal{R} \Vdash^p \mathrm{S}(\phi)]$	(viii)	$\neg[\mathcal{R} \Vdash^p \mathrm{W}(\phi)]$

where ϕ is an arbitrary formula and P is an arbitrary variable. For (v - viii) consider any $D \subseteq A$ where both D and $A - D$ are dense in A, and let P be true on $D \cup (1, \infty)$.

4.5 Show that for each transitive (monomodal) structure $\mathcal{A}$ both

$$\mathcal{A} \Vdash^u \Box\Diamond^2\phi \leftrightarrow \Box\Diamond\phi \quad , \quad \mathcal{A} \Vdash^u (\Box\Diamond)^2\phi \leftrightarrow \Box\Diamond\phi$$

hold for all formulas ϕ.

4.6 Since our modal language contains the constants $\bot$ and $\top$, it is possible to construct formulas which contain no variables. Such a formula is called a *sentence*. For instance, in the monomodal case, the sentences are

$$\top$$

$$\Diamond\top \qquad \Box\top$$

$$\Diamond\Diamond\top \quad \Box\Diamond\top \quad \Diamond\Box\top \quad \Box\Box\top$$

$$\vdots \quad \vdots \quad \vdots$$

$$\vdots \qquad\qquad \vdots$$

$$\vdots \quad \vdots \quad \vdots$$

$$\Diamond\Diamond\bot \quad \Box\Diamond\bot \quad \Diamond\Box\bot \quad \Box\Box\bot$$

$$\Diamond\bot \qquad \Box\bot$$

$$\bot$$

together with all the boolean and modal combinations of these. Sentences do not need a valuation in order to acquire a truth value. Thus, given a sentence ϕ, a structure $\mathcal{A}$, and an element a, if

$$a \Vdash \phi$$

holds for some valuation on $\mathcal{A}$ then it holds for all valuations on $\mathcal{A}$.

Sentences can be used to capture some information about chains in a structure.

Let us restrict ourselves to the monomodal case, and let $\mathcal{A}$ be a given structure. For each $n \in \mathsf{N}$ and elements $a, b \in A$ let

$$a \xrightarrow{n} b \tag{4.2}$$

mean there is a chain

$$a = a_0 \longrightarrow a_1 \longrightarrow \cdots \longrightarrow a_n = b$$

of length n between a and b. In particular, $a \xrightarrow{0} a$ holds vacuously. Let

$$a \xrightarrow{n} \checkmark \quad , \quad a \xrightarrow{n} \times$$

mean, respectively,

$$\text{there is some } b \quad , \quad \text{there is no } b$$

for which (4.2) holds. It is also useful to say

$$\begin{array}{rcl} a \text{ can see } b & \Leftrightarrow & a \longrightarrow b \\ a \text{ is blind} & \Leftrightarrow & a \longrightarrow \times. \end{array}$$

(The confusion between the use of '$\xrightarrow{n}$' as an iterated transition and a labelled transition is quite deliberate.)

(a) Show that

(i) $a \Vdash \Diamond^n \top \Leftrightarrow a \xrightarrow{n} \checkmark$

(ii) $a \Vdash \Box^n \bot \Leftrightarrow a \xrightarrow{n} \times$.

Under what conditions do

$$a \Vdash \Box^n \top \quad , \quad a \Vdash \Diamond^n \bot$$

hold?

(b) Show that

(i) $a \Vdash \Box\Diamond\bot \Leftrightarrow a$ is blind

(ii) $a \Vdash \Diamond\Box\bot \Leftrightarrow a$ can see a blind element

(iii) $a \Vdash \Box\Diamond\top \Leftrightarrow a$ can see no blind elements

(iv) $a \Vdash \Diamond\Box\top \Leftrightarrow a$ is not blind

hold.

(c) Find sentences which express the following.

(i) Every element seen by a is blind.

(ii) The element a can see a non-blind element.

(iii) Every element seen by a is either blind or can see a non-blind element.

(iv) The element a can see a non-blind element which can see only blind elements.

4.7 We say a set Φ of sentences is *independent* if for each $\phi \in \Phi$ there is a structure $\mathcal{A}$ and an element a of $\mathcal{A}$ such that

- for each $\psi \in (\Phi - \{\phi\})$, $\quad a \Vdash \psi$
- $a \Vdash \neg\phi$.

This exercise exhibits an infinite independent set of sentences.

For each $m \in \mathsf{N}$ let $W(m)$ be the sentence

$$\Diamond^{m+2}\top \rightarrow \Box^{m+1}\Diamond\top.$$

Let $\mathcal{A}_m$ be the structure with $2m+4$ elements $a, \ldots, b, \ldots, c$ where

$$a \xrightarrow{m+1} b \quad , \quad a \xrightarrow{m+2} c$$

and where the chains between a and b and a and c are disjoint apart from the common source a.

(a) Draw a picture of $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$, and $\mathcal{A}_3$.

(b) Show that for each $l < m < n$ both

$$a \Vdash \Box^{l+1}\Diamond\top \quad , \quad a \Vdash \Box^{n+2}\bot$$

hold in $\mathcal{A}_m$.

(c) Show that in $\mathcal{A}_m$

$$a \Vdash \neg W(m)$$

holds, whereas

$$a \Vdash W(k)$$

holds for all $k \neq m$.

(d) Show that the set $\{W(m) \mid m \in \mathsf{N}\}$ is independent.

4.8 Consider the monomodal structure $\mathcal{N} = (\mathsf{N}, \longrightarrow)$ where

$$x \longrightarrow y \Leftrightarrow x \leq y + 1$$

for all $x, y \in \mathsf{N}$. This is called the *recession structure*. To determine the truth values of formulas in $\mathcal{N}$ observe that each $X \subseteq \mathsf{N}$ falls into exactly one of the following five types.

(1) $X = \mathsf{N}$.

(2) There is some $a \in \mathsf{N}$ with $[a+1, \infty] \subseteq X \subseteq \{a\}'$ (where $(\cdot)'$ is the complement operation.

(3) There is some $a \in X$ with $X \subseteq [0, a]$.

(4) $X = \emptyset$.

(5) X is neither finite nor cofinite.

Note that types (2,3) are complementary, as are types (1,4). Type (5) is self complementary.

(a) Verify the following tables of values for these five types.

X	$\Box x$	$\Diamond X$	$\Box^2 X$	$\Diamond^2 X$	$\Box\Diamond X$	$\Diamond\Box X$
(1)	N	N	N	N	N	N
(2)	$[a+2, \infty]$	N	$[a+3, \infty]$	N	N	N
(3)	$\emptyset$	$[0, a+i]$	$\emptyset$	$[0, a+i]$	$\emptyset$	$\emptyset$
(4)	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
(5)	$\emptyset$	N	$\emptyset$	N	N	$\emptyset$

(b) Determine which of the standard formulas are modelled by $\mathcal{N}$.

(c) Which of the following formulas

(i) $\Box(\Box\phi \rightarrow \Box\psi) \vee \Box(\Box\psi \rightarrow \Box\phi)$
(ii) $\Box(\Box^2\phi \rightarrow \Box^3\phi) \rightarrow (\Box\phi \rightarrow \Box^2\phi)$
(iii) $\Box(\Box(\phi \rightarrow \Box\phi) \rightarrow \Box^3\phi) \rightarrow \Box^4\phi$

are modelled by $\mathcal{N}$?

4.9 Consider two structures $\mathcal{A}$ and $\mathcal{B}$ of the same signature. We say $\mathcal{B}$ is a *substructure* of $\mathcal{A}$ and write

$$\mathcal{B} \subseteq \mathcal{A} \tag{4.3}$$

if $B \subseteq A$ and the transition relations of $\mathcal{B}$ are the restrictions of the corresponding relations of $\mathcal{A}$, i.e. for each label i and elements $b, y \in B$

$$b \xrightarrow{i} y \text{ holds in } \mathcal{B} \Leftrightarrow b \xrightarrow{i} y \text{ holds in } \mathcal{A}.$$

Note that every non-empty subset of A is the carrier of a substructure of $\mathcal{A}$.

Given (4.3), each valuation α on $\mathcal{A}$ restricts to a valuation β on $\mathcal{B}$ defined by

$$b \in \beta(P) \Leftrightarrow b \in \alpha(P)$$

for each variable P and $b \in B$. We then say that the valued structure $(\mathcal{B}, \beta)$ is a substructure of $(\mathcal{A}, \alpha)$ and write

$$(\mathcal{B}, \beta) \subseteq (\mathcal{A}, \alpha) \tag{4.4}$$

We say $\mathcal{B}$ or $(\mathcal{B}, \beta)$ is a *generated substructure* of $\mathcal{A}$ or $(\mathcal{A}, \alpha)$ and write

$$\mathcal{B} \subseteq_g \mathcal{A} \quad \text{or} \quad (\mathcal{B}, \beta) \subseteq_g (\mathcal{A}, \alpha)$$

if the appropriate one of (4.3) or (4.4) holds and, for each label i and elements $b \in B$ and $a \in A$,

$$b \xrightarrow{i} a \Rightarrow a \in B$$

holds.

(a) Show that for a given valued structure $(\mathcal{A}, \alpha)$

(i) each non-empty subset B of A carries a substructure $(\mathcal{B}, \beta)$ of $\mathcal{A}$,

(ii) for each element $a \in A$ there is a smallest generated substructure $\mathcal{B} = \mathcal{A}(a)$ of $\mathcal{A}$ which contains a.

(b) Hence show that if $(\mathcal{B}, \beta) \subseteq_g (\mathcal{A}, \alpha)$ then

$$(\mathcal{B}, \beta, b) \Vdash \phi \Leftrightarrow (\mathcal{A}, \alpha, b) \Vdash \phi$$

for all $b \in B$ and formulas ϕ.

(c) For an arbitrary formula ϕ let $\{\phi\}^*$ be the set of all formulas $[\mathrm{i}]\phi$ for a compound label i. Show that for each valued structure $(\mathcal{A}, \alpha)$ and element a of $\mathcal{A}$, the equivalence

$$(\mathcal{A}, \alpha, a) \Vdash^p \{\phi\}^* \Leftrightarrow (\mathcal{B}, \beta) \Vdash^v \phi$$

holds where $\mathcal{B} = \mathcal{A}(a)$.

4.10 Let $\mathcal{A}$ be any structure of an arbitrary signature, and let $A^\vee$ be the set of ultrafilters $p, q, r, \ldots$ on A. (Recall that an ultrafilter on A is a set p of subsets of A with certain appropriate closure properties.) For each $a \in A$ let

$$a^\vee = \{X \in \mathcal{P}A \mid a \in X\}$$

be the principal ultrafilter on a.

For each label i let $\xrightarrow{i}$ be the relation on $A^\vee$ given by

$$p \xrightarrow{i} q \Leftrightarrow (\forall X \in \mathcal{P}A)[[i]X \in p \Rightarrow X \in q]$$

for $p, q \in a^\vee$, where $[i]$ is the modal operation on $\mathcal{P}A$ corresponding to the label i.

This gives us a structure $\mathcal{A}^\vee$ based on $A^\vee$ which is called the *ultrafilter extension* of the structure $\mathcal{A}$.

For a given valuation α on $\mathcal{A}$ let $\alpha^\vee$ be the valuation on $\mathcal{A}^\vee$ given by

$$p \in \alpha^\vee(P) \Leftrightarrow \alpha(P) \in p$$

for each $p \in A^\vee$ and variable P. We compare the two valued structures $(\mathcal{A}, \alpha)$ and $(\mathcal{A}^\vee, \alpha^\vee)$.

(a) Show that

$$a \overset{i}{\longrightarrow} b \Leftrightarrow a^\vee \overset{i}{\longrightarrow} b^\vee$$

for each label i and elements $a, b \in A$.

(b) Show that

$$(\mathcal{A}^\vee, \alpha^\vee, p) \Vdash \phi \Leftrightarrow [\![\phi]\!] \in p$$

for each formula ϕ and $p \in A^\vee$.

(c) Show that

(i) $(\mathcal{A}, \alpha, a)^\vee \Vdash^p \phi \Leftrightarrow (\mathcal{A}, \alpha, a) \Vdash^p \phi$

(ii) $(\mathcal{A}, \alpha)^\vee \Vdash^v \phi \Leftrightarrow (\mathcal{A}, \alpha) \Vdash^v \phi$

(iii) $\mathcal{A}^\vee \Vdash^u \phi \Rightarrow \mathcal{A} \Vdash^u \phi$ for each formula ϕ and $a \in A$.

[These results form a modal version of Łos's result from first order model theory.]

4.11 Let $\mathcal{N} = (\mathsf{N}, \longrightarrow)$ be the monomodal structure where

$$a \longrightarrow b \Leftrightarrow a < b$$

for all $a, b \in \mathsf{N}$ and consider the ultrafilter extension $\mathcal{N}^\vee$ of $\mathcal{N}$. Show the following.

(a) For each $x \subseteq \mathsf{N}$ and $p \in \mathsf{N}^\vee$ if $\Box X \in p$ then X is cofinite.

(b) For all $p, q \in \mathsf{N}^\vee$,

$$p \longrightarrow q$$

holds whenever q is non-principal.

(c) The structure $\mathcal{N}^\vee$ has some reflexive points.

(d) There are formulas ϕ such that both

$$\mathcal{N} \Vdash^u \phi \quad , \quad \neg[\mathcal{N}^\vee \Vdash^u \phi]$$

hold.

4.12 In the following chapters we show that many properties of transition structures can be captured by sets of modal formulas. However, not all such properties can be captured in this way. For instance let us say a monomodal structure $\mathcal{A}$ is *good* if it is transitive, serial, and each of its elements can see a reflexive element.

(a) Show that the structure $\mathcal{N}$ of Exercise 4.11 is transitive and serial but not good, whereas its ultrafilter extension $\mathcal{N}^\vee$ is good.

(b) Show that there is no set of formulas Γ such that a structure models Γ precisely when it is good.