Chapter 5

Correspondence theory

5.1 Introduction

In Chapter 3 I claimed that modal logic should be viewed as a tool for describing and analysing properties of structures (that is, of labelled transition structures). How can this be, and how effective is this tool? The kind of simple things we might want to know about a relation are whether it is reflexive, symmetric, transitive, confluent, etc. We may want to know whether one relation is included in another, or is the converse of another, or whether one relation can be decomposed as the composite of two others, etc. We may want to know more complicated things like whether a relation is well-founded, or whether one relation is the $*$-closure of another.

Remarkably, these and many other properties are characterized by quite simple modal formulas. It is this characterizing ability which makes modal logic such a powerful tool. Once it is understood, it can be seen that modal logic is a quite extensive part of full second order logic, and it is the ability to capture second order properties which gives it is power.

5.2 Some examples

As an illustration of the kind of thing we are going to do we begin with a quite simple example of a correspondence result. In this result we focus on one particular label with its associated relation $\prec$ and connective $\Box$.

5.1 PROPOSITION. *For each structure $\mathcal{A}$ the conditions*

- *(i) The distinguished relation $\prec$ is reflexive.*
- *(ii) For each formula ϕ, $\mathcal{A} \Vdash^u \Box\phi \rightarrow \phi$.*
- *(iii) For some variable P, $\mathcal{A} \Vdash^u \Box P \rightarrow P$.*

are equivalent.

Before we prove this result, we make a few remarks. There are many correspondence results, and all have the same form as Proposition 5.1. A structural property (i) is shown to be equivalent to the modelling of a certain family of formulas (ii). These formulas are all the substitution instances of a certain set of basic formulas (iii). Thus the implication (ii) $\Rightarrow$ (iii) is trivial, and (iii) $\Rightarrow$ (ii) is an application of Proposition 4.13. The implication (i) $\Rightarrow$ (ii) is always proved by direct verification (and is no harder than the implication (i) $\Rightarrow$ (iii)). The implication (iii) $\Rightarrow$ (i) follows by the use of a particular valuation chosen in a suitable way. The choice of this valuation is the only non-routine part of the proof.

Proof of Proposition 5.1. (i) $\Rightarrow$ (ii). This is Lemma 4.6 (i).
(ii) $\Rightarrow$ (iii). This is trivial
(iii) $\Rightarrow$ (i). With the variable P given by (iii), for a fixed element a, consider any valuation α such that for each $x \in A$,

$$x \Vdash P \quad \Leftrightarrow \quad x \prec a$$

holds. Then (as in Chapter 4, Section 4.3) we have

$$a \Vdash \Box p$$

and hence, invoking (iii), we obtain $a \Vdash P$. Thus $a \prec a$, as required. ■

Our second example has a very similar proof.

5.2 PROPOSITION. *For each structure $\mathcal{A}$ the conditions*

(i) The distinguished relation $\prec$ is transitive.
(ii) For each formula ϕ, $\mathcal{A} \Vdash^u \Box\phi \to \Box^2\phi$.
(iii) For some variable P, $\mathcal{A} \Vdash^u \Box P \to \Box^2 P$.

are equivalent.

Proof. (i) $\Rightarrow$ (ii). This is Lemma 4.6 (ii).
(ii) $\Rightarrow$ (iii). This is trivial.
(iii) $\Rightarrow$ (i). With the variable P given by (iii), for a fixed element a, consider any valuation α such that for each $x \in A$,

$$x \Vdash P \Leftrightarrow x \prec a.$$

Then $a \Vdash \Box P$ so that, invoking (iii), we have $a \Vdash \Box^2 P$. Thus, for all elements b and c,

$$c \prec b \prec a \quad \Rightarrow \quad c \Vdash P \quad \Rightarrow \quad c \prec a$$

which gives (i). ■

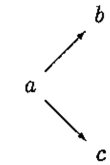
It isn't always the same valuation which is required for the proof of (iii) $\Rightarrow$ (i). For instance, consider the following.

5.3 PROPOSITION. *For each structure $\mathcal{A}$ the conditions*

(i) The distinguished relation $\prec$ is deterministic.
(ii) For each formula ϕ, $\mathcal{A} \Vdash^u \Diamond\phi \to \Box\phi$.
(iii) For some variable P, $\mathcal{A} \Vdash^u \Diamond P \to \Box P$.

are equivalent.

Proof. (i) $\Rightarrow$ (ii). This left as an exercise.
(ii) $\Rightarrow$ (iii). This is trivial.
(iii) $\Rightarrow$ (i). Consider any elements a, b, c with



and, with the variable given by (iii), consider any valuation such that for each $x \in A$,

$$x \Vdash P \Leftrightarrow x = b.$$

Then $a \Vdash \Diamond P$ so that (iii) gives $a \Vdash \Box P$ and hence $c \Vdash P$, i.e. $c = b$, as required. ■

There is a whole family of results of this kind with virtually the same proof. For instance all of the following properties of a structure $\mathcal{A}$ and label i fall into this class.

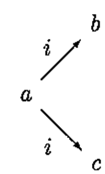
(a) $\mathcal{A}$ is i-serial, i.e. for each $a \in A$ there is some $b \in A$ with $b \prec_i a$.

(b) $\mathcal{A}$ is i-reflexive, i.e. the relation $\prec_i$ is reflexive.

(c) $\mathcal{A}$ is i-symmetric.

(d) $\mathcal{A}$ is i-transitive.

(e) $\mathcal{A}$ is i-euclidean, i.e. for each divergent wedge



we have $b \prec_i c$ (and $c \prec_i b$).

(f) $\mathcal{A}$ is i-pathetic.

(g) $\mathcal{A}$ is i-deterministic.

(h) $\mathcal{A}$ is i-dense.

For each of these properties you should attempt to prove the correspondence result. Of course, the problem is to find the characterizing formula, but once this has been done the proof is virtually routine.

5.3 The confluence property

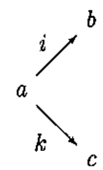
There is a single result which covers all the correspondence results of the last section and many more as well. We begin our discussion of this result here and will return to it later (in Chapter 6).

We need some terminology.

Fix the labels

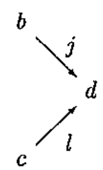
$$i, j, k, l.$$

These may be distinct or have repetitions among them. We say a structure $\mathcal{A}$ has the (i, j, k, l)-*confluence property* if for each divergent wedge of elements



(5.1)

there is a convergent wedge



(5.2)

This property subsumes all the properties mentioned in Section 5.2, but before we see how this comes about let us state and prove the correspondence result for confluence.

5.4 THEOREM. *For each structure $\mathcal{A}$ the conditions*

(i) $\mathcal{A}$ has (i, j, k, l)-confluence.
(ii) For each formula ϕ, $\mathcal{A} \Vdash^u \langle i\rangle [j]\phi \rightarrow [k]\langle l\rangle\phi$.
(iii) For some variable P, $\mathcal{A} \Vdash^u \langle i\rangle [j]P \rightarrow [k]\langle l\rangle P$.

are equivalent.

Proof. (i) $\Rightarrow$ (ii). Suppose $\mathcal{A}$ has the confluence property and that

$$a \Vdash \langle i\rangle [j]\phi$$

for some element a, formula ϕ, and valuation on $\mathcal{A}$. This hypothesis gives some $b \prec_i a$ with $b \Vdash [j]\phi$. We are required to verify that $a \Vdash [k]\langle l\rangle\phi$, so consider any element $c \prec_k a$. We then have a wedge (5.1), so the confluence property provides an element d with

$$d \prec_j b \quad , \quad d \prec_l c.$$

From the first of these we get $d \Vdash \phi$, and hence the second gives $c \Vdash \langle l\rangle\phi$ which is enough to complete the proof.

(ii) $\Rightarrow$ (iii). This is trivial.

(iii) $\Rightarrow$ (i). Consider any given wedge (5.1) and, with the variable given by (iii), consider any valuation such that

$$x \Vdash P \Leftrightarrow x \prec_j b$$

(for all $x \in A$). Then $b \Vdash [j]P$ and hence $a \Vdash \langle i\rangle [j]P$ so that, invoking (iii), we have $c \Vdash \langle l\rangle P$ and hence there is some $d \prec_l c$ with $d \Vdash P$. This last condition ensures that $d \prec_j b$, and so we have the required wedge (5.2). ■

How does this result cover all the correspondence results of the last section, and how does the confluence property generalize the properties (a)–(h) of that section? Let us look at some cases.

(a) For a given relation $\prec$ (of the structure $\mathcal{A}$), choose the labels j and l so that $\prec_j$, $\prec_l$, and $\prec$ agree. Let i and k label equality, i.e. for each $x, y \in A$

$$y \prec_i x \Leftrightarrow x = y \Leftrightarrow y \prec_k x.$$

A divergent wedge (5.1) is then a triple of elements

$$b = a = c$$

i.e. an arbitrary element a. A convergent wedge is then given by an element d such that

$$d \prec b = a \quad , \quad d \prec c = a$$

so that confluence reduces to seriality. Note also that the corresponding formula of Theorem 5.4 becomes

$$\Box\phi \rightarrow \Diamond\phi$$

as expected.

(b) Let j and k label equality. Then the confluence property asserts that $\prec_i$ is a subrelation of $\prec_l$. In particular when $\prec_i$ also labels equality, this says that $\prec_l$ is reflexive.

(c) Let i and j label equality. Then the confluence property says that $\prec_k$ is included in the converse of $\prec_l$, i.e.

$$y \prec_k x \quad \Rightarrow \quad x \prec_l y.$$

In particular, when k and l both label $\prec$, this says that $\prec$ is symmetric.

(d) See if you can work this out for yourself, but be warned, there is a slight catch here.

(e) Let i, j, and k label $\prec$ and let l label equality. Then the confluence property says that $\prec$ is euclidean. The corresponding shape of formula is

$$\Diamond\Box\phi \to \Box\phi$$

which, by taking the contrapositive, is equivalent to the shape

$$\Diamond\phi \to \Box\Diamond\phi.$$

Alternatively we could let i, k, and l label $\prec$ and let j label equality.

You should work out the remaining cases for yourself. Also, for each particular case, it is instructive to go through the proof of Theorem 5.4 for that case. After a few of these you will begin to see what is going on.

5.4 Some non-confluence properties

It is not the case that all correspondence results are covered by Theorem 5.4; some of them are quite a bit more complicated. In this section we look at a couple of results which illustrate some of these possible extra complications.

For the first one it is convenient to have some terminology. Thus, for want of a better word, let us say a relation $\prec$ of a structure $\mathcal{A}$ is *tree-like* if

$$b \prec a \text{ and } c \prec a \quad \Rightarrow \quad b \prec c \text{ or } c \prec b$$

for all $a, b, c \in A$. The characterization of this property illustrates how more than one variable may be needed.

5.5 PROPOSITION. *For each structure $\mathcal{A}$ the conditions*

(i) The distinguished relation $\prec$ is tree-like.
(ii) For all formulas ϕ, ψ, $\mathcal{A} \Vdash^u \Box(\Box\phi \to \psi) \vee \Box(\Box\psi \to \phi)$.
(iii) For some pair P, Q of distinct variables, $\mathcal{A} \Vdash^u \Box(\Box P \to Q) \vee \Box(\Box$

are equivalent.

Proof. (i) $\Rightarrow$ (ii). Suppose that $\prec$ is tree-like and that

$$\text{not}[a \Vdash \Box(\Box\phi \to \psi)]$$

for some element a, formulas ϕ and ψ, and valuation on $\mathcal{A}$. Then

$$a \Vdash \Diamond(\Box\phi \wedge \neg\psi)$$

so there is some $b \prec a$ with

$$b \Vdash \Box\phi \quad , \quad b \Vdash \neg\psi.$$

Consider also any element $c \prec a$ with

$$c \Vdash \Box\psi.$$

We require that $c \Vdash \phi$.

Since $\prec$ is tree-like we know that either

$$b \prec c \quad \text{or} \quad c \prec b.$$

If the first of these holds then $b \Vdash \psi$, which we know is not so. Thus $c \prec b$ and hence $c \Vdash \phi$, as required.

(ii) $\Rightarrow$ (iii). This is trivial

(iii) $\Rightarrow$ (i). Suppose that $b \prec a$ and $c \prec a$, and, with the variables given by (iii) consider any valuation for which

$$x \Vdash P \Leftrightarrow x \prec b \quad , \quad x \Vdash Q \Leftrightarrow x \prec c$$

(for $x \in A$). This is possible since P and Q are distinct. By construction we have

$$b \Vdash \Box P \quad , \quad c \Vdash \Box Q$$

and we know, by the hypothesis, that either

$$a \Vdash \Box(\Box P \to Q) \quad \text{or} \quad a \Vdash \Box(\Box Q \to P).$$

If the first of these holds then $b \Vdash \Box P \to Q$ so that $b \Vdash Q$ and hence $b \prec c$. If the second holds then, by a similar argument $c \prec b$. ■

So far all the structural properties we have characterized have been elementary (in the sense that they are first order definable). The power of modal logic comes from its ability to deal with some non-elementary properties. We now give an example of such a property.

Before we give the characterization we need a preliminary result.

5.6 LEMMA. *For a structure $\mathcal{A}$ and variable P suppose that*

$$\mathcal{A} \Vdash^u \Box(\Box P \to P) \to \Box P.$$

Then the corresponding transition relation $\longrightarrow$ is transitive.

Proof. We make use of Proposition 4.5. Thus, consider a valuation such that for each $x \in A$,

$$x \Vdash P \Leftrightarrow (\forall y)[y \prec\!* x \Rightarrow y \prec a]$$

where a is a fixed element and $\prec\!*$ is the $*$-closure of $\prec$. Then $a \Vdash \Box(\Box P \to P)$ and hence, invoking the hypothesis, we have $a \Vdash \Box P$. But then, for all $b, c \in A$, we have

$$c \prec b \prec a \Rightarrow c \prec b \Vdash P \Rightarrow c \prec a$$

which gives the required result. ■

A distinguished relation $\longrightarrow$ of a structure $\mathcal{A}$ is said to be *well-founded* if there is no sequence $(a_r | r < w)$ of elements with

$$a_0 \longrightarrow a_1 \longrightarrow \cdots \longrightarrow a_r \longrightarrow \cdots \qquad (r < w).$$

This property, even when combined with transitivity, is not elementary, hence the interest of the following result.

5.7 THEOREM. *For each structure $\mathcal{A}$ the conditions*

(i) The distinguished relation $\to$ is transitive and well-founded.
(ii) For each formula ϕ, $\mathcal{A} \Vdash^u \Box(\Box\phi \to \phi) \to \Box\phi$.
(iii) For some variable P, $\mathcal{A} \Vdash^u \Box(\Box P \to P) \to \Box P$.

are equivalent.

Proof. (i) $\Rightarrow$ (ii). Suppose $\prec$ is well-founded and consider any element a, formula ϕ, and valuation with

$$a \Vdash \Box(\Box\phi \to \phi).$$

Consider also, any element b with

$$b \prec a \quad , \quad b \Vdash \neg\phi. \qquad (5.3)$$

Then, from the position of a, we have $b \Vdash \neg\Box\phi$, which gives some element

$$c \prec b \quad \text{with} \quad c \Vdash \neg\phi.$$

Since $\prec$ is transitive this gives

$$c \prec a \quad \text{and} \quad c \Vdash \neg\phi.$$

Hence, by iterating this construction, we obtain a sequence of elements $(b_r | r < w)$ with

$$b = b_0 \longrightarrow b_1 \longrightarrow \cdots \longrightarrow b_r \longrightarrow \cdots \qquad (r < w)$$

and $b_r \Vdash \neg\phi$ for all $r < w$. Since well-foundedness obstructs such a sequence, we see there can be no initial element b satisfying (5.3). Thus $a \Vdash \neg\Diamond\neg\phi$, i.e. $a \Vdash \Box\phi$, as required.

(ii) $\Rightarrow$ (iii). This is trivial.

(iii) $\Rightarrow$ (i). Suppose (iii) holds. Then, by Lemma 5.6, the relation $\prec$ is transitive, so we must show that $\prec$ is well-founded.

By way of contradiction suppose there is a sequence $(a_r | r < w)$ with

$$a_0 \longrightarrow a_1 \longrightarrow \cdots \longrightarrow a_r \longrightarrow \cdots \qquad (r < w).$$

With the variable P given by (iii) consider any valuation such that, for $x \in A$,

$$x \Vdash \neg P \Leftrightarrow (\exists r < w)[x = a_r].$$

In particular, $a_1 \Vdash \neg P$ so that $a_0 \Vdash \Diamond\neg P$ and hence, invoking (iii), we have

$$a_0 \Vdash \Diamond(\Box P \wedge \neg P).$$

This gives some $x \prec a$ with

$$x \Vdash \Box P \quad \text{and} \quad x \Vdash \neg P.$$

The second of these ensures that $x = a_r$ for some $r < w$. But then, by the first we have

$$a_{r+1} \Vdash P$$

which is the required contradiction. ■

5.5 Exercises

5.1 Consider a language with not necessarily distinct labels i, j, k, l. Below is a list of pairs of a formula shape (s) depending on an arbitrary formula ϕ, and a structural property (p). For each of these pairs, show that a structure models shape (s) precisely when it has property (p).

(a) (s) $\langle i \rangle\phi \to \phi$
(p) i-pathetic.

(b) (s) $\langle i\rangle\phi \to \langle j\rangle\phi$
(p) The relation $\xrightarrow{i}$ is included in the relation $\xrightarrow{j}$.

(c) (s) $\langle i\rangle\phi \to [j]\phi$
(p) All wedges of the form

$$\begin{array}{ccc} a & \xrightarrow{i} & b \\ {\scriptstyle j}\downarrow & & \\ c & & \end{array}$$

have $c = b$.

(d) (s) $\langle i\rangle\phi \to \langle j\rangle[k]\phi$
(p) For each pair $a \xrightarrow{i} b$ there is some c with $a \xrightarrow{j} c$ such that the only x with $c \xrightarrow{k} x$ is $x = b$.

(e) (s) $\langle i\rangle\phi \to [j]\langle k\rangle\phi$
(p) All wedges of the form

$$\begin{array}{ccc} a & \xrightarrow{i} & b \\ {\scriptstyle j}\downarrow & & \\ c & & \end{array}$$

have $c \xrightarrow{k} b$.

(f) (s) $\langle i\rangle\phi \to \langle j\rangle[k]\langle l\rangle\phi$
(p) For each pair $a \xrightarrow{i} b$ there is some c with $a \xrightarrow{j} c$ such that for all d, if $c \xrightarrow{k} d$ then $d \xrightarrow{l} b$.

(g) (s) $\langle i\rangle\phi \to [j]\langle k\rangle[l]\phi$
(p) For each wedge

$$\begin{array}{ccc} a & \xrightarrow{i} & b \\ {\scriptstyle j}\downarrow & & \\ c & & \end{array}$$

there is some element d with $c \xrightarrow{k} d$ such that the only x with $d \xrightarrow{l} x$ is $x = b$.

5.2 Let i, j, k, l, m, n be fixed labels. For each of the following pairs, show that a structure models the shape (s) precisely when it has property (p).

(a) (s) $\langle i\rangle[j]\phi \to [k]\langle l\rangle[m]\phi$.
(p) For each wedge of elements

$$\begin{array}{ccc} a & \xrightarrow{i} & b \\ {\scriptstyle k}\downarrow & & \\ c & & \end{array}$$

there is some element d with $c \xrightarrow{l} d$ such that

$$d \xrightarrow{m} x \Rightarrow b \xrightarrow{j} x$$

holds for all elements x.

(b) (s) $\langle i\rangle[j]\phi \to \langle k\rangle[l]\langle m\rangle\phi$.
(p) For each transition $a \xrightarrow{i} b$, there is some transition $a \xrightarrow{k} c$ such that for each transition $c \xrightarrow{l} d$, there is a wedge

$$\begin{array}{ccc} & & b \\ & & \downarrow{\scriptstyle j} \\ d & \xrightarrow[m]{} & e \end{array}.$$

(c) (s) $\langle i\rangle[j]\phi \to [k]\langle l\rangle[m]\langle n\rangle\phi$.
(p) For each wedge

$$\begin{array}{ccc} a & \xrightarrow{i} & b \\ {\scriptstyle k}\downarrow & & \\ c & & \end{array}$$

there is a transition $c \xrightarrow{l} d$ such that for each transition $d \xrightarrow{m} e$

$$\begin{array}{ccc} & & b \\ & & \downarrow{\scriptstyle j} \\ e & \xrightarrow[n]{} & f \end{array}.$$

5.3 Consider a modal language with labels i, j, k, l, m, n (where these need be neither distinct nor atomic). Let $K(i, j, k, l, m, n)$ be the generalized K-shape

$$[i]([j]\phi \to [k]\psi) \to [l]([m]\phi \to [n]\psi)$$

for arbitrary ϕ and ψ. Show that a structure $\mathcal{A}$ models $K(i, j, k, l, m, n)$ if and only if for all elements a, b, c with

$$a \xrightarrow{l} b \xrightarrow{n} c$$

there is some element d such that

- $a \xrightarrow{i} d \xrightarrow{k} c$
- for all elements x, $\quad d \xrightarrow{j} x \;\Rightarrow\; b \xrightarrow{m} x$

hold.

5.4 Not all correspondence results need to be proved by chosing a suitable valuation. For instance let k and l be fixed natural numbers with $k > l$. Show that for each transitive (monomodal) structure $\mathcal{A}$, the three conditions:

(i) For all formulas ϕ, $\quad \mathcal{A} \Vdash^{p} \Box^{k}\Diamond\Box\phi \to \Box^{l}\Diamond\Box\phi$.

(ii) $\mathcal{A} \Vdash^{u} \Diamond^{k}\Box\bot \vee \Box^{l}\Diamond\top$.

(iii) For each $a \in A$ one of

- There is some blind b with $a \xrightarrow{k} b$.
- There is no blind b with $a \xrightarrow{l} b$.

holds.

are equivalent.

5.5 Let k and l be fixed natural numbers. Show that for each transitive (monomodal) structure $\mathcal{A}$, the four conditions:

(i) For all formulas ϕ, $\quad \mathcal{A} \Vdash^{u} \Box^{k}\Diamond\phi \to \Diamond^{l}\Box\Diamond\phi$.

(ii) For all formulas ϕ, $\quad \mathcal{A} \Vdash^{u} \Box^{k}\Diamond\Box\phi \to \Diamond^{l}\Box\Diamond\phi$.

(iii) $\mathcal{A} \Vdash^{u} \Diamond^{k}\Box\bot \vee \Diamond^{l}\Box\Diamond\top$.

(iv) For each $a \in A$, there is some $b \in A$ such that one of

- $a \xrightarrow{k} b$ and b is blind
- $a \xrightarrow{l} b$ and no element seen by b is blind

holds.

are equivalent.

5.6 The results of Exercise 5.1 can be generalized. To do this let us say a *modal operator* M is a sequence of $\langle i\rangle$ and $\boxed{i}$ for varying labels i. For instance

$$\emptyset,\ \langle i\rangle,\ \boxed{i},\ \langle i\rangle\boxed{i},\ \boxed{j}\langle i\rangle,\ \langle i\rangle\boxed{i}\langle k\rangle,\ \ldots$$

are all modal operators. These operators may be defined recursively by:

- $\emptyset$ is a modal operator;

- if M is a modal operator then so are

$$\langle i\rangle \mathrm{M} \quad , \quad \boxed{i}\mathrm{M}$$

for each label i.

Note that for each modal operator M and formula ϕ, the compound $\mathrm{M}\phi$ is also a formula. Structural properties corresponding to the shapes

$$\langle i\rangle\phi \to \mathrm{M}\phi$$

can be developed, but this requires some preliminary notation.

Fix a structure $\mathcal{A}$. For each modal operator M a relation

$$-\{\mathrm{M}\}\!\!\rightarrow$$

is recursively defined on $\mathcal{A}$ using the following clauses.

For each pair $a, b \in A$

$$a -\{\emptyset\}\!\!\rightarrow b \;\Leftrightarrow\; a = b$$

$$a -\{\langle i\rangle\mathrm{M}\}\!\!\rightarrow b \;\Leftrightarrow\; \text{There is some } x \text{ with } \quad a \xrightarrow{i} x -\{\mathrm{M}\}\!\!\rightarrow b$$

$$a -\{\boxed{i}\mathrm{M}\}\!\!\rightarrow b \;\Leftrightarrow\; \text{For all } x, \quad a \xrightarrow{i} x \;\Rightarrow\; x -\{\mathrm{M}\}\!\!\rightarrow b$$

(for arbitrary i and M).

(a) Give explicit descriptions of the relations

$$-\{\langle i\rangle\}\!\!\rightarrow \qquad -\{\boxed{i}\}\!\!\rightarrow$$

$$-\{\langle i\rangle\boxed{j}\}\!\!\rightarrow \qquad -\{\boxed{i}\langle j\rangle\}\!\!\rightarrow$$

$$-\{\langle i\rangle\boxed{j}\langle k\rangle\}\!\!\rightarrow \quad -\{\boxed{i}\langle j\rangle\boxed{k}\}\!\!\rightarrow$$

on $\mathcal{A}$.

(b) For a fixed element b and variable P, let β be any valuation such that $\beta(P) = \{b\}$. Show that

$$a -\{\mathrm{M}\}\!\!\rightarrow b \;\Leftrightarrow\; (\mathcal{A}, \beta, a) \Vdash^{p} \mathrm{M}P$$

holds for all $a \in A$ and modal operators M.

(c) Show that for all modal operators M, and pairs a, b with $a -\{\mathrm{M}\}\!\!\rightarrow b$, the implication

$$b \Vdash \phi \;\Rightarrow\; a \Vdash \mathrm{M}\phi$$

holds for all valuations and formulas ϕ.

(d) Show that $\mathcal{A}$ models the shape

$$\langle i \rangle \phi \to \mathrm{M}\phi$$

if and only if the relation $\overset{i}{\longrightarrow}$ is included in the relation $-\{\mathrm{M}\}\!\!\to$.

(e) Show how the results of Exercise (5.1) are particular cases of (d).

5.7 The results of this chapter and the previous exercises show that many formula shapes are equivalent to structural properties which are describable in elementary terms. This is not the case for all formula shapes. The simplest example of a non-elementary shape is given by the *McKinsey* formula

$$\mathrm{M}(\phi) \;:=\; \Box\Diamond\phi \to \Diamond\Box\phi$$

on an arbitrary formula ϕ. The class of models of this shape is quite weird, in particular the following result forms a basis of a proof that the class is not elementary.

Let S be a fixed countably infinite set and, as usual, let $2 = \{0,1\}$. Let $[S \longrightarrow 2]$ be the set of functions

$$f : S \longrightarrow 2.$$

For each such function f let $\neg f$ be the complementary function given by

$$(\neg f)(x) = 1 - f(x)$$

(for $x \in S$). Let F be any subset of $[S \longrightarrow 2]$. We use this to construct a transition structure $\mathcal{A}(F)$.

Thus set

$$A(F) \;=\; \{a\} \cup S \cup (S \times 2) \cup F$$

where a is some new element. Let $\longrightarrow$ be the transition relation on $A(F)$ such that

$$a \longrightarrow x \longrightarrow (x,i) \longrightarrow (x,i) \qquad a \longrightarrow f \longrightarrow (x, f(x))$$

for all $x \in S, i \in 2, f \in F$, with no other transitions holding. Let

$$\mathcal{A}(F) \;=\; (A(F), \longrightarrow).$$

(a) Show that for each $u \in A(F)$:

(i) $u = a \;\Leftrightarrow\;$ there is no $v \in A(F)$ with $v \longrightarrow u$
(ii) $u \in S \times 2 \;\Leftrightarrow\; u \longrightarrow u$
(iii) $u \in S \;\Leftrightarrow\;$ there are precisely two $v \in A(F)$ with $u \longrightarrow v$
(iv) $u \in F \;\Leftrightarrow\;$ there are at least three $v \in A(F)$ with $u \longrightarrow v$

(b) Show that

(i) $(x,i) \Vdash \phi \leftrightarrow \Box\phi$ (ii) $(x,i) \Vdash \phi \leftrightarrow \Diamond\phi$
(iii) $x \Vdash \mathrm{M}(\phi)$ (iv) $x \Vdash \Box\Diamond\phi \to \Box^2\phi$
(v) $f \Vdash \mathrm{M}(\phi)$ (vi) $f \Vdash \Box\Diamond\phi \to \Box^2\phi$

for all appropriate x, i, f and ϕ, and valuations on $\mathcal{A}(F)$.

(c) For arbitrary $g : S \longrightarrow 2$ with $\neg g \notin F$ consider the valuation γ on $\mathcal{A}(F)$ where, for some variable P,

$$y \Vdash P \;\Leftrightarrow\; (\exists x \in S)[y = (x, g(x))]$$

for each $y \in A(F)$.

(i) Show that $a \Vdash \Box\Diamond P$.

(ii) Hence show that if $\mathcal{A}(F)$ models $\mathrm{M}(P)$ then $g \in F$.

(d) Suppose the set is closed under $\neg(\cdot)$. Show that $\mathcal{A}(F)$ models the shape M if and only if $F = [S \longrightarrow 2]$.

5.8 Consider the following choice principle.

(∗) Suppose that $\longrightarrow$ is a transitive relation on the set X such that

$$(\forall x \in X)(\exists y \in X)[x \longrightarrow y \;\wedge\; x \neq y].$$

Then there are sets Y, Z with

$$Y \cap Z = \emptyset \quad , \quad Y \cup Z = X$$

and such that

$$(\forall x \in X)(\exists y \in Y, z \in Z)[x \longrightarrow y \;\wedge\; x \longrightarrow z]$$

holds.

This is a version of the Axiom of Choice. In some restricted situations non-elementary properties can become elementary.

(a) Using (∗), show that a transitive structure $\mathcal{A} = (A, \longrightarrow)$ models McKinsey's axiom if and only if: For each $a \in A$ there is some $b \in A$ with $a \longrightarrow b$ and such that

$$b \longrightarrow x \;\Rightarrow\; x = b$$

holds for all $x \in A$.

(b) Can you prove (∗)?