Chapter 8

Standard formal systems

8.1 Introduction

As I explained in Chapter 1, the principal aim of (non-modal) propositional logic is to give a syntactic description of the semantic consequence relation $\models$. This is done by setting up a formal system controlling a proof theoretic consequence relation $\vdash$ whose operational properties mimic (we hope) those of $\models$. The success of this programme culminates in the proof of the completeness theorem asserting that

$$\Phi \vdash \phi \qquad \Leftrightarrow \qquad \Phi \models \phi$$

for appropriate sets of formulas Φ and formulas ϕ.

We now wish to carry out a similar programme for modal logic. This is not entirely straight forward for, as we saw in Chapter 7, there are many different modal semantic consequence relations, and it is not at all clear which of these we should isolate for proof theoretic analysis.

The honest approach to this is to state at the outset which semantic consequence relations we are interested in, and then design an appropriate formal system. We won't do this. Our approach (which, in fact, is the usual approach) will be to first construct a proof theoretic consequence relation (or rather, a whole class of such consequence relations, all of a similar kind), and then see which, if any, of the semantic consequence relations have been captured.

A second difference between modal and modal-free logic has a deeper significance. In the non-modal case there are several different possible styles of proof systems (Hilbert, Natural, Sequence, ...). One of the important achievements of propositional logic (and some of its non-modal enrichments) is that these different styles are intertranslatable. (The deduction property and the cut elimination property are two of the tools used in these translations.)

In modal logic the situation is much more delicate; there are significant technical problems (some of which have not yet been solved) to be faced when translating one style into another.

We step over these problems by choosing a proof style which best suits our purpose, namely a Hilbert style of system.

For us a formal system S with its associated consequence relation $\vdash_S$ is determined by the following data.

- An acceptable set $\mathcal{S}$ of logical axioms.
- A set $\mathcal{R}$ of rules of inference.

The precise definition of these notions will be given later, but once they have been fixed we may construct the associated consequence relation

$$\Phi \vdash_S \phi$$

between sets of formulas Φ and formulas ϕ in a standard way. Thus this relation holds precisely when there is a witnessing formal deduction consisting of a finite sequence

$$\phi_0, \phi_1, \ldots, \phi_n$$

of formulas generated from Φ using S and finishing with the formula ϕ.

Each formal system S is determined by its two components $\mathcal{S}$ (the axioms) and $\mathcal{R}$ (the rules). We will fix the rules once and for all so, for us, the system S is determined solely by the choice of axioms $\mathcal{S}$.

By its very nature a proof theoretic system is concerned with syntactic manipulations in a certain formal language. It is thus important that we know what is a part of that language and what are merely convenient devices for talking about the language. Therefore, for clarity, I will restate the meaning of 'the modal language of signature I'.

8.1 CONVENTION. *For an index set I, the modal language of signature I has as its basic symbols $\bot, \top, \wedge, \vee, \rightarrow, \neg$, and $[i]$ for each $i \in I$. (In particular, $\leftrightarrow$ and $\langle i \rangle$ are not part of the language.)* ■

8.2 Formal systems defined

We need to make precise the following notions.

- An acceptable set of axioms $\mathcal{S}$.
- The rules of inference $\mathcal{R}$.
- A witnessing deduction $\phi_0, \phi_1, \ldots, \phi_n$.

This we now do.

8.2 DEFINITION. An acceptable set of axioms is a set $\mathcal{S}$ of formulas which

- contains all (modal-free) tautologies,
- contains all formulas of the shape

$$\text{(K)} \qquad [i](\theta \rightarrow \psi) \rightarrow ([i]\theta \rightarrow [i]\psi)$$

for all formulas θ, ψ and labels i,

- is closed under substitution. ■

Each set $\mathcal{A}$ of formulas generates a smallest acceptable set of axioms $\mathcal{S}$. Namely, let $\mathcal{A}^+$ be the set formed by adding to $\mathcal{A}$ all tautologies and all formulas (K) and then let $\mathcal{S}$ be the set of substitution instances of $\mathcal{A}^+$. This gives us a convenient way of describing particular sets of axioms, for we need only give a generating set $\mathcal{A}$. There are a host of examples of such sets, many of which are formed from various combinations of the shapes D, T, B, ... of Chapter 2. We look at some of these examples in the next section.

8.3 DEFINITION. The rules of inference are modus ponens

$$\text{(MP)} \qquad \frac{\theta \quad \theta \rightarrow \phi}{\phi}$$

and necessitation

$$\text{(Ni)} \qquad \frac{\phi}{[i]\phi}$$

for each label $i \in I$. ■

The meaning of these rules will become clear after we have described the notion of a formal deduction.

8.4 DEFINITION. Let $\mathcal{S}$ be the set of axioms determining a given formal system S and let Φ be an arbitrary set of formulas (the hypothesis set).

(a) A witnessing S-deduction from Φ is a finite sequence

$$\phi_0, \phi_1, \ldots, \phi_n$$

of formulas such that for each index $0 \leq r \leq n$ one of the following holds.

(hyp) The formula ϕ_r is an hypothesis, i.e. a member of Φ.

(ax) The formula ϕ_r is an axiom, (i.e. a member of $\mathcal{S}$).

(mp) The formula ϕ_r is obtained by (MP) from two earlier formulas, i.e. there are indexes $t, s < r$ with

$$\phi_t = (\phi_s \rightarrow \phi_r).$$

(n) The formula ϕ_r is obtained by (Ni) from an earlier formula, i.e. there is an index $s < r$ with

$$\phi_r = [i]\phi_s$$

(for some label $i \in I$).

(b) A formula ϕ is an S-consequence of Φ

$$\Phi \vdash_{\mathsf{S}} \phi$$

if there is a witnessing S-deduction from Φ whose final term is ϕ. ■

A better understanding of these notions will be obtained by looking at some particular examples of formal deductions. For this purpose several such examples are given in the next section. For the time being let us look at some box (and diamond) manipulations.

8.5 LEMMA. *For each formal system* S *and formulas* ψ *and* ϕ,

$$\vdash_{\mathsf{S}} \psi \to \phi \quad \Rightarrow \quad \vdash_{\mathsf{S}} \Box\psi \to \Box\phi$$

holds.

Proof. Let us abbreviate $\psi \to \phi$ by θ. Then any witnessing deduction for θ can be extended to one for $\Box\psi \to \Box\phi$ as follows.

$$\begin{array}{cr} \vdots & \\ \theta & (Hyp) \\ \Box\theta & (N) \\ \Box\theta \to (\Box\psi \to \Box\phi) & (K) \\ \Box\psi \to \Box\phi & (MP) \end{array}$$

Down the right hand side I have indicated the justification for each of the terms in the deduction. ■

A useful consequence of this is concerned with equivalences.

8.6 COROLLARY. *For all formulas* ψ, ϕ

$$\vdash_{\mathsf{S}} \psi \leftrightarrow \phi \quad \Rightarrow \quad \vdash_{\mathsf{S}} \Box\psi \leftrightarrow \Box\phi$$

holds. In particular, if ψ *and* ϕ *are tautologically equivalent then*

$$\vdash_{\mathsf{S}} \Box\psi \leftrightarrow \Box\phi$$

holds.

Recall that the diamond connective $\Diamond$ has been introduced by

$$\Diamond\phi \quad \equiv \quad \neg\Box\neg\phi.$$

Thus, using the sequence of tautological equivalences

$$\neg\Diamond\neg\phi \leftrightarrow \neg\neg\Box\neg\neg\phi \leftrightarrow \Box\neg\neg\phi \leftrightarrow \Box\phi$$

we have

$$\vdash_{\mathsf{S}} \Box\phi \leftrightarrow \neg\Diamond\neg\phi.$$

Similarly, by taking contrapositives, the above Corollary gives

$$\vdash_{\mathsf{S}} \psi \leftrightarrow \phi \quad \Rightarrow \quad \vdash_{\mathsf{S}} \Diamond\psi \leftrightarrow \Diamond\phi.$$

This shows that for most purposes $\Diamond$ can be regarded as a primitive symbol.

We conclude this section with a couple of observations.

There is some considerable confusion in the literature over the correct notion of a proof theoretic consequence relation. Most authors use a weaker version which can be described in various ways but which amounts to the following notion.

8.7 DEFINITION. For a set of formulas Φ and formula ϕ let

$$\Phi \vdash^{w}_{\mathsf{S}} \phi$$

mean there is some finite part $\phi_1, \dots, \phi_n$ of Φ with

$$\vdash_{\mathsf{S}} \phi_1 \wedge \ldots \wedge \phi_n \to \phi$$

(where here the hypothesis set is empty). ■

It is an easy exercise (which you should do) to show that

$$\Phi \vdash^{w}_{\mathsf{S}} \phi \quad \Rightarrow \quad \Phi \vdash_{\mathsf{S}} \phi$$

holds, however, in general, the converse is false. Note, however, that when used with an empty hypothesis set, the two relations agree, i.e.

$$\vdash^{w}_{\mathsf{S}} \phi \quad \Leftrightarrow \quad \vdash_{\mathsf{S}} \phi$$

holds for all formulas ϕ.

The important difference between $\vdash^{w}_{\mathsf{S}}$ and $\vdash_{\mathsf{S}}$ is the Deduction Property. By construction we have

$$\Phi, \theta \vdash^{w}_{\mathsf{S}} \phi \quad \Rightarrow \quad \Phi \vdash^{w}_{\mathsf{S}} (\theta \to \phi)$$

but, in general, this is not true of $\vdash_{\mathsf{S}}$. To see this note that for any formula ϕ, the rule (N) gives

$$\phi \vdash_{\mathsf{S}} \Box\phi.$$

but only for special systems S (the pathetic systems) do we have

$$\vdash_{\mathsf{S}} (\phi \to \Box\phi).$$

It is precisely this 'defect' of $\vdash_{\mathsf{S}}$ which leads many people to consider only the weaker version $\vdash^{w}_{\mathsf{S}}$.

The second observation concerns the monotonicity of this consequence notion. We state the relevant result but leave the proof as an exercise.

8.8 **PROPOSITION.** *For each pair of sets of axioms $\mathcal{S}$ and $\mathcal{T}$ with $\mathcal{S} \subseteq \mathcal{T}$ and for each pair of hypothesis sets Φ and Ψ with $\Phi \subseteq \Psi$, the implication*

$$\Phi \vdash_{\mathsf{S}} \phi \quad \Rightarrow \quad \Psi \vdash_{\mathsf{T}} \phi$$

holds for all formulas ϕ.

8.3 Some monomodal systems

In this section we look at some particular, and well known, examples of formal systems formulated in the monomodal language, i.e. the language with just one label. All of these examples are generated using various combinations of the standard shapes of formulas D, T, B, 4, 5 (as given in Chapter 2).

The first and smallest formal system K is the one whose axioms are all instances of tautologies together with all instances of the shape K. (Thus the set of axioms of K is the smallest allowed by Definition 8.2.)

Larger systems can be formed by extending the set of axioms. Thus, for instance, let

$$\mathsf{KD} \;,\; \mathsf{KT} \;,\; \mathsf{KB} \;,\; \mathsf{K4} \;,\; \mathsf{K5}$$

be the systems whose axioms comprise the smallest acceptable set containing the formulas

$$\mathrm{D} \;,\; \mathrm{T} \;,\; \mathrm{B} \;,\; 4 \;,\; 5$$

respectively. Similarly let

KDT ,	KDB ,	KD4 ,	KD5
	KTB ,	KT4 ,	KT5
		KB4 ,	KB5
			K45

be the systems whose axioms comprise the smallest acceptable set containing the indicated shapes. Continuing further we may form such systems as

$$\mathsf{KDTB} \;,\; \mathsf{KDB4} \;,\; \mathsf{KB45}$$

etc.

Two of these systems are particularly important and have given names. These are

$$\mathsf{S4} = \mathsf{KT4} \quad , \quad \mathsf{S5} = \mathsf{KT5}.$$

On the face of it the five shapes D, T, B, 4, and 5 give us $2^5 = 32$ different systems, however, as we will see, not all of these are distinct. There are, in fact, only 15 such systems.

How do we compare two systems and what do we mean by two systems being 'the same'? We take a pragmatic, extensional view of this.

Thus given two systems S and T (based on the sets of axioms $\mathcal{S}$ and $\mathcal{T}$) we write

$$\mathsf{S} \le \mathsf{T}$$

if

$$\vdash_{\mathsf{S}} \phi \quad \Rightarrow \quad \vdash_{\mathsf{T}} \phi$$

holds for all formulas ϕ. It can be seen that $\mathsf{S} \le \mathsf{T}$ implies the apparently stronger property that

$$\Phi \vdash_{\mathsf{S}} \phi \quad \Rightarrow \quad \Phi \vdash_{\mathsf{T}} \phi$$

holds for all hypothesis sets Φ and formulas ϕ.

We now agree to say that S and T are the same if both the comparisons

$$\mathsf{S} \le \mathsf{T} \quad \text{and} \quad \mathsf{T} \le \mathsf{S}$$

hold.

Using this notation let us look at some of the comparisons which hold between $\mathsf{K}, \mathsf{KD}, \mathsf{KT}, \ldots, \mathsf{KDTB45}$. These comparisons will follow from various examples of witnessing deductions.

Our first example shows how the shape D is captured by T. For an arbitrary formula ϕ let

$$\alpha = \Box\neg\phi \quad , \quad \beta = \Box\phi \quad , \quad \gamma = \neg\alpha = \Diamond\phi$$

(so that D(ϕ) is $\beta \to \gamma$, T(ϕ) is $\beta \to \phi$, and T($\neg\phi$) is $\alpha \to \neg\phi$). The following sequence is a witnessing S-deduction for any system S whose axioms include the shape T. The justification for each formula is given on the right.

$$\begin{array}{cl}
\alpha \to \neg\phi & (T) \\
(\alpha \to \neg\phi) \to (\phi \to \neg\alpha) & (Taut) \\
\phi \to \gamma & (MP) \\
\beta \to \phi & (T) \\
(\beta \to \phi) \to ((\phi \to \gamma) \to (\beta \to \gamma)) & (Taut) \\
(\phi \to \gamma) \to (\beta \to \gamma) & (MP) \\
\beta \to \gamma & (MP)
\end{array}$$

This deduction immediately gives the following.

8.9 LEMMA. $\mathsf{KD} \le \mathsf{KT}$, $\mathsf{KD4} \le \mathsf{S4}$, $\mathsf{KD5} \le \mathsf{S5}$.

The next example shows how **S5** captures the shape B. Some of the terms have been omitted and you should fill in these positions for yourself.

$$\begin{array}{cl}
\Box\neg\phi \to \neg\phi & (T) \\
\vdots & (Taut) \\
\phi \to \Diamond\phi & (MP) \\
\Diamond\phi \to \Box\Diamond\phi & (5) \\
\vdots & (Taut) \\
\vdots & (MP) \\
\phi \to \Box\Diamond\phi & (MP)
\end{array}$$

Combining this with the previous Lemma we get the following.

8.10 LEMMA. $\mathsf{KDB5} \le \mathsf{S5}$.

All modal systems have the following two useful derived rules of inference.

$$\frac{\theta \to \phi}{\Box\theta \to \Box\phi} \quad , \quad \frac{\theta \to \phi}{\Diamond\theta \to \Diamond\phi}$$

We refer to these jointly as (EN), i.e. as 'extended necessitation'. The first of these rules is justified by

$$\begin{array}{cl}
\theta \to \phi & (Hyp) \\
\Box(\theta \to \phi) & (N) \\
\Box(\theta \to \phi) \to (\Box\theta \to \Box\phi) & (K) \\
\Box\theta \to \Box\phi & (MP)
\end{array}$$

and the second by

$$\begin{array}{c}
\theta \to \phi \\
\vdots \\
\neg\phi \to \neg\theta \\
\vdots \\
\Box\neg\phi \to \Box\neg\theta \\
\vdots \\
\Diamond\theta \to \Diamond\phi
\end{array}$$

These derived rules make some deductions easier to display. For instance, working in any extension of **KB5** we have

$$\begin{array}{cl}
\Diamond\neg\phi \to \Box\Diamond\neg\phi & (5) \\
\vdots & \\
\Diamond\Box\phi \to \Box\phi & \\
\vdots & (EN) \\
\Box\Diamond\Box\phi \to \Box\Box\phi & \\
\Box\phi \to \Box\Diamond\Box\phi & (B) \\
\vdots & \\
\Box\phi \to \Box\Box\phi &
\end{array}$$

which immediately gives the following.

8.11 LEMMA. $\mathsf{K4} \le \mathsf{KB5}$, $\mathsf{KDB4} \le \mathsf{KDB5}$.

The previous example showed how, in the presence of B, the shape 4 can be captured by the shape 5. We can now do the reverse.

$$\begin{array}{cl}
\Box\neg\phi \to \Box^2\neg\phi & (4) \\
\vdots & \\
\Diamond^2\phi \to \Diamond\phi & \\
\vdots & (EN) \\
\Box\Diamond^2\phi \to \Box\Diamond\phi & \\
\Diamond\phi \to \Box\Diamond^2\phi & (B) \\
\vdots & \\
\Diamond\phi \to \Box\Diamond\phi &
\end{array}$$

As a consequence of this we have the following.

8.12 LEMMA. $\mathsf{K5} \le \mathsf{KB4}$, $\mathsf{KT5} \le \mathsf{KTB4}$.

Finally we show how to capture the shape T.

$$\begin{array}{cl}
\Box\phi \to \Box^2\phi & (4) \\
\Box^2\phi \to \Diamond\Box\phi & (D) \\
\vdots & \\
\Box\phi \to \Diamond\Box\phi & \\
\neg\phi \to \Box\Diamond\phi & (B) \\
\vdots & \\
\Diamond\Box\phi \to \phi & \\
\vdots & \\
\Box\phi \to \phi &
\end{array}$$

Combining this with several of the previous results gives us two different axiomatizations of **S5**.

8.13 THEOREM. S5 = KDB4 = KDB5.

Proof. We have

$$
\begin{array}{rcll}
\mathsf{S5} & = & \mathsf{KT5} & \\
 & \leq & \mathsf{KTB4} & \text{(by Lemma 8.12)} \\
 & \leq & \mathsf{KDB4} & \text{(from above)} \\
 & \leq & \mathsf{KDB5} & \text{(by Lemma 8.11)} \\
 & \leq & \mathsf{S5} & \text{(by Lemma 8.10)}
\end{array}
$$

as required. ■

8.4 Some polymodal systems

Historically the monomodal systems of the previous section were developed for various philosophical reasons, for instance, as attempts to formalize some of the properties of the modalities 'is necessary', 'is obligatory', 'is known', etc. Alongside these developments there were also various analyses of the properties of tenses (in natural languages) and time. This brought forth tense logic which has now become *temporal logic* and which also encompasses a much wider field of applicability.

Temporal logic is a the study of certain bimodal system (and various enrichments of these) designed to capture the flow of time. The associated transition structures have the form

$$\mathcal{A} = (A, \xrightarrow{+}, \xrightarrow{-})$$

where the two carried transition relations are the forward passage and the backwards passage through time. The two relations are not unconnected, however the minimal restrictions we need to put on them are as follows.

8.14 DEFINITION. A *temporal structure* is a bimodal structure

$$\mathcal{A} = (A, \xrightarrow{+}, \xrightarrow{-})$$

where each of the transition relations is transitive, and each is the converse of the other. ■

Using some of the correspondence results that we have obtained we see that such structures can be isolated by a standard formal system. Thus consider the bimodal language with box operators $[-]$ and $[+]$, and in this language consider all the formulas of the following shapes.

$$
\begin{array}{rcl}
[+]\phi & \rightarrow & [+]^2\phi \\
[-]\phi & \rightarrow & [-]^2\phi \\
\phi & \rightarrow & [+]\langle - \rangle\phi \\
\phi & \rightarrow & [-]\langle + \rangle\phi
\end{array}
$$

Let TEMP be the standard formal system axiomatized by the formulas of these above four shapes.

The following result is straight forward.

8.15 THEOREM. *A bimodal structure $\mathcal{A}$ (as above) is a temporal structure precisely when it models the system* TEMP.

Proof. The first two axioms ensure that the two relations are transitive, the third ensures that $\xrightarrow{+}$ is included in the converse of $\xrightarrow{-}$, and the fourth ensures that $\xrightarrow{-}$ is included in the converse of $\xrightarrow{+}$. ■

As remarked already, temporal logic can be used to analyse some of the tense properties of natural languages. The recent development of situation theory is an attempt to analyse the more general information content of natural languages. This has thrown up another bimodal system.

Thus consider the bimodal language with box operators $[\approx]$ and $\Box$ and let SL (situation logic) be the formal system whose axioms are all the formulas

$$
\begin{array}{lcl}
[\approx]\phi \rightarrow \phi & \qquad & \Box\phi \rightarrow \phi \\
[\approx]\phi \rightarrow [\approx][\approx]\phi & & \Box\phi \rightarrow \Box\Box\phi \\
\phi \rightarrow [\approx]\langle\approx\rangle\phi & &
\end{array}
$$

and the formulas

$$[\approx]\Box\phi \rightarrow \Box[\approx]\phi$$

for arbitrary formulas ϕ. Structures for this language have the form

$$\mathcal{A} = (A, \xrightarrow{\approx}, \longrightarrow)$$

and the models of SL are easily characterized.

8.16 THEOREM. *A structure $\mathcal{A}$ (as above) models* SL *precisely when the three conditions*

- *the relation $\xrightarrow{\approx}$ is an equivalence*
- *the relation $\longrightarrow$ is a pre-ordering*
- *for each configuration $a \longrightarrow b \xrightarrow{\approx} c$ there is an element d with $a \xrightarrow{\approx} d \longrightarrow c$*

hold.

Proof. A routine application of various correspondence results. ■

Dynamic logic is a naturally occurring polymodal logic. Furthermore in this logic the set of labels has its own algebraic structure and this leads to some quite intricate properties. At brief discussion of this logic is given in Chapter 14.

8.5 Soundness properties

In Chapter 7 we introduced three semantic consequence relations $\models^k$ (for $k = u, v, p$). These made no reference to any underlying basis of 'logically valid' modal formulas. We can now correct this omission.

8.17 **DEFINITION.** Let S be a standard formal system with set of axioms $\mathcal{S}$. Let k be a kind. For each set of formulas Φ and formula ϕ, the relation

$$\Phi \models^k_{\mathsf{S}} \phi$$

holds precisely when each k-structure which is a model of S and of Φ is also a model of ϕ. ■

Note that

$$\Phi \models^k_{\mathsf{S}} \phi \quad \Leftrightarrow \quad \Phi \cup \mathcal{S} \models^k \phi$$

so that an analysis of $\models^k_{\mathsf{S}}$ could be reduced to one of $\models^k$. However, the parameterized version $\models^k_{\mathsf{S}}$ leads to a much richer theory.

We have now attached to each formal system S five consequence relations; the two proof theoretic relations

$$\vdash^w_{\mathsf{S}} \quad , \quad \vdash_{\mathsf{S}}$$

and the three semantic relations

$$\models^p_{\mathsf{S}} \quad , \quad \models^v_{\mathsf{S}} \quad , \quad \models^u_{\mathsf{S}} .$$

How do these relations interact? We have already observed that

$$\Phi \vdash^w_{\mathsf{S}} \phi \quad \Rightarrow \quad \Phi \vdash_{\mathsf{S}} \phi$$

and a simple application of Proposition 7.2 (with $\Psi = \Phi \cup \mathcal{S}$) gives

$$\Phi \models^p_{\mathsf{S}} \phi \quad \Rightarrow \quad \Phi \models^v_{\mathsf{S}} \phi \quad \Rightarrow \quad \Phi \models^u_{\mathsf{S}} \phi.$$

We can also add to this a soundness result.

8.18 **THEOREM.** *For each formal system* S*, hypothesis set* Φ*, and formula* ϕ*, the implication*

$$\Phi \vdash_{\mathsf{S}} \phi \quad \Rightarrow \quad \Phi \models^v_{\mathsf{S}} \phi$$

holds.

Proof. This is proved by induction on the deduction witnessing $\Phi \vdash_{\mathsf{S}} \phi$. The base cases and the induction steps follow by an application of Lemma 7.3 ■

This result can be used to show that different sets of axioms give different formal systems. For instance, by Lemma 8.9 we know that KD $\leq$ KT. Thus to show that the two systems are not the same it suffices to find a formula ϕ with

$$\vdash_{\mathsf{KT}} \phi \quad , \quad \text{not}[\vdash_{\mathsf{KD}} \phi]$$

and, by the soundness result, the second of these can be justified by showing

$$\text{not}[\models^v_{\mathsf{KD}} \phi].$$

To this end, for an arbitrary variable P let

$$\phi = \mathrm{T}(P) = (\Box P \to P).$$

Trivially $\vdash_{\mathsf{KT}} \phi$.

Now consider any structure $\mathcal{A}$ with just two points a and b and accessibility relation

$$a \longrightarrow b \longrightarrow a$$

i.e. for all points x

$$x \prec a \quad \Leftrightarrow \quad x = b \quad , \quad x \prec b \quad \Leftrightarrow \quad x = a.$$

Note that $\mathcal{A}$ is serial and hence is a model of KD.

Consider any valuation α on $\mathcal{A}$ with

$$\alpha(P) = \{b\}.$$

Then $(\mathcal{A}, \alpha)$ models KD. Also

$$a \Vdash \neg P \quad , \quad b \Vdash P$$

so that

$$a \Vdash \neg P \quad , \quad a \Vdash \Box P$$

and hence

$$a \Vdash \neg\phi.$$

Thus $(\mathcal{A}, \alpha)$ does not model ϕ which gives the required result.

In the next chapter we prove a couple of completeness results; one applicable to all standard formal systems, the other applicable only to a restricted class. For both results we do not deal with the full power of the appropriate consequence relation, but only with the 'logical' case, i.e. we restrict to the case where the hypothesis set Φ is empty.

8.6 Exercises

8.1 Verify that

(i) $\vdash_{\mathsf{K}} \Box(\theta \wedge \psi) \leftrightarrow \Box\theta \wedge \Box\psi$

(ii) $\vdash_{\mathsf{K}} \Diamond(\theta \to \psi) \to (\Box\theta \to \Diamond\psi)$

(iii) $\vdash_{\mathsf{K}} \Diamond\top \to \mathrm{D}(\phi)$

(iv) $\vdash_{\mathsf{K}} \neg\Diamond\theta \to \Box(\theta \to \phi)$

(v) $\vdash_{\mathsf{K}} \Box\phi \to \Box(\theta \to \phi)$

(vi) $\vdash_{\mathsf{K}} (\Diamond\theta \to \Box\phi) \to \Box(\theta \to \phi)$

(vii) $\vdash_{\mathsf{K}} (\Diamond\theta \to \Box\phi) \to (\Box\theta \to \Box\phi)$

(viii) $\vdash_{\mathsf{K}} (\Diamond\theta \to \Box\phi) \to (\Diamond\theta \to \Diamond\phi)$

(ix) $\vdash_{\mathsf{K}} \Diamond\theta \to \mathrm{D}(\phi)$

for arbitrary formulas θ, ψ and ϕ.

8.2 Show that

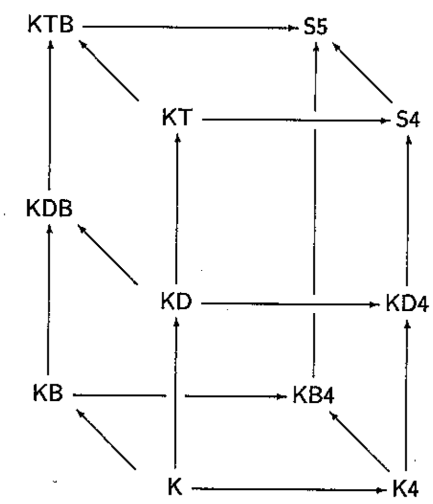
(i) $\vdash_{\mathsf{K4}} \Box\Diamond^2\phi \leftrightarrow \Box\Diamond\phi$

(ii) $\vdash_{\mathsf{K4}} (\Box\Diamond)^2\phi \leftrightarrow \Box\Diamond\phi$

and hence verify the results of Exercise 4.5.

8.3 For the four shapes of formulas D,T,B,4 consider the $16 = 2^4$ possible extensions of the system K obtained by adding some of these shapes as axioms.

(a) Show that this gives no more than 11 different systems with inclusions as shown.



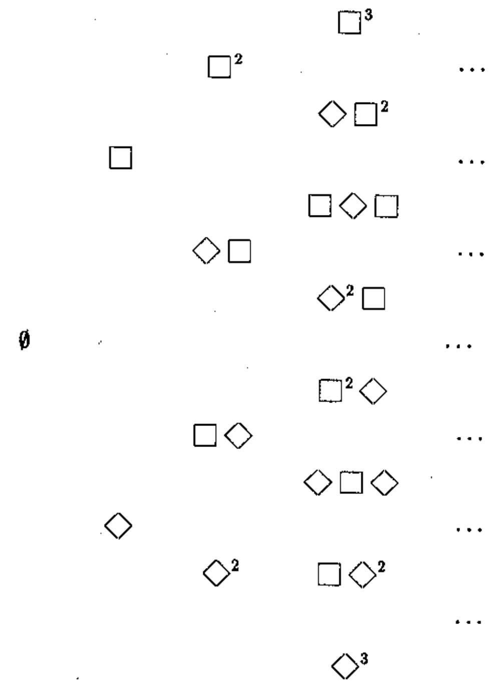
(b) By considering structures with no more than three elements, show that these 11 systems are distinct.

8.4 For each of the systems S of Exercise 8.3, consider the system S′ formed by adding 5 as a further axiom. Show that this produces no more than four new systems

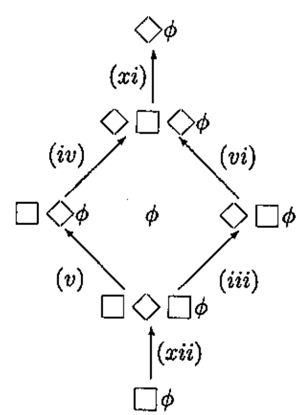
K5, K45, KD5, KD45

and fit these into the diagram of Exercise 8.3.

8.5 For a given formula ϕ in a monomodal language, the *modal variants* of ϕ are the formulas $\mathrm{M}\phi$ where M is a modal operator of the form



etc. In general these are all distinct and there no implications between them. However, in $\mathsf{S4} = \mathsf{KT4}$ each formula has no more than 7 modal variants. These with some implications between them are shown in the following diagram.



You are invited to verify these implications and to insert the two missing ones. To do this let $\vdash$ be $\vdash_{\mathsf{S4}}$ and prove the following.

(a) Use T and (EN), and finally 4 to show that

(i)	$\vdash \Box\phi \to \phi$	(ii)	$\vdash \phi \to \Diamond\phi$
(iii)	$\vdash \Box\Diamond\Box\phi \to \Diamond\Box\phi$	(iv)	$\vdash \Box\Diamond\phi \to \Diamond\Box\Diamond\phi$
(v)	$\vdash \Box\Diamond\Box\phi \to \Box\Diamond\phi$	(vi)	$\vdash \Diamond\Box\phi \to \Diamond\Box\Diamond\phi$
(vii)	$\vdash \Box\Diamond\phi \to \Diamond\phi$	(viii)	$\vdash \Box\phi \to \Diamond\Box\phi$
(ix)	$\vdash \Diamond\Box\Diamond\phi \to \Diamond^2\phi$	(x)	$\vdash \Box^2\phi \to \Box\Diamond\Box\phi$
(xi)	$\vdash \Diamond\Box\Diamond\phi \to \Diamond\phi$	(xii)	$\vdash \Box\phi \to \Box\Diamond\Box\phi$

hold for all formulas ϕ.

(b) Setting $\phi := \Diamond\phi$ in (a) and using T and 4, show that

$$\vdash (\Box\Diamond)^2\phi \leftrightarrow \Box\Diamond\phi \quad , \quad \vdash (\Diamond\Box)^2\phi \leftrightarrow \Diamond\Box\phi$$

and hence each sequence of four or more modal operators collapses to three or fewer.

(c) By considering suitable models of $\mathsf{S4}$, show that there is at least one formula ϕ for which the above 11 variants are distinct.

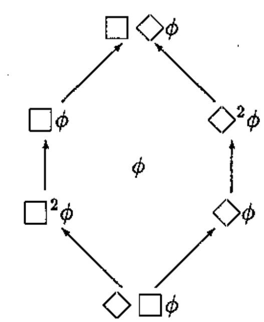
8.6 Consider the formal system $\mathsf{K5}$ and let $\vdash$ be $\vdash_{\mathsf{K5}}$.

(a) Verify that

(i)	$\vdash \Diamond\Box\phi \to \Box\phi$	(ii)	$\vdash \Box\Diamond\Box\phi \to \Box^2\phi$
(iii)	$\vdash \Diamond^2\Box\phi \to \Diamond\Box\phi$	(iv)	$\vdash \Diamond\Box\phi \to \Box\Diamond\Box\phi$
(v)	$\vdash \Diamond\Box^2\phi \to \Box^2\phi$	(vi)	$\vdash \Diamond\Box\phi \to \Box^2\phi$
(vii)	$\vdash \Box\Diamond\Box\phi \to \Box^3\phi$	(viii)	$\vdash \Diamond\Box\phi \to \Box^3\phi$
(ix)	$\vdash \Box^2\phi \to \Box\Diamond\Box\phi$	(x)	$\vdash \Diamond\Box\phi \to \Diamond^2\Box\phi$
(xi)	$\vdash \Box^2\phi \to \Box^3\phi$	(xii)	$\vdash \Diamond\Box\phi \to \Diamond\Box^2\phi$

hold for all formulas ϕ.

(b) Show that in $\mathsf{K5}$ the modal variants of a formula ϕ are arranged as



and these formulas can be distinct.

8.7 Show that in S5 each variable has precisely three modal variants and describe how these are arranged.

8.8 Let $\vdash$ be $\vdash_{\mathsf{KTB}}$. Show that for each variable P, formula ϕ and integers $m \geq n \geq 0$, both

$$\vdash \Box^m \phi \to \Box^n \phi \quad , \quad \neg[\vdash \Box^n P \to \Box^{n+1} P]$$

hold, and hence P has infinitely many modal variants in KTB.

8.9 Describe the modal variants of a variable P in K45 and in KB4.

8.10 Find temporal structures which do *not* model the following shapes.

(i) $[+]\bot$ (ii) $\langle + \rangle \top$ (iii) $[+]\phi \to \phi$ (iv) $\phi \to$ [illegible]

(v) $[-]^2\phi \to [+]\phi$ (vi) $[+]^2\phi \to [+]\phi$ (vii) $\langle + \rangle\phi \to [+]\phi$ (viii) $[+]\phi$ [illegible]

8.11 Which of the following are modelled by all temporal structures?

(i) $\langle + \rangle [+]\phi \to [+]\langle + \rangle\phi$ (ii) $\langle + \rangle [+]\phi \to [-]\langle - \rangle\phi$

(iii) $\langle + \rangle [-]\phi \to [-]\langle + \rangle\phi$ (iv) $\langle + \rangle [-]\phi \to [+]\langle - \rangle\phi$

8.12 The basic temporal system TEMP is too weak to capture many of the assumed properties of the passage of time. In this and the next two exercises we consider some suitable strengthenings of TEMP. For each formula ϕ let

$$[\approx]\phi \quad \text{abbreviate} \quad [-]\phi \wedge \phi \wedge [+]\phi$$

and let LINTIM be the extension of TEMP formed by the addition of the two axioms

$$[\approx]\phi \to [-][+]\phi \quad , \quad [\approx]\phi \to [+][-]\phi .$$

To understand the import of this, for each temporal structure $\mathcal{A}$, let $\approx$ be the relation on $\mathcal{A}$ given by

$$a \approx b \Leftrightarrow a \xrightarrow{-} b \text{ or } a = b \text{ or } a \xrightarrow{+} b$$

(for $a, b \in A$).

(a) Show that for each model $\mathcal{A}$ of TEMP

$$a \Vdash \phi \Leftrightarrow \begin{cases} \text{For all } x, \\ a \approx x \Rightarrow x \Vdash \phi \end{cases}$$

holds for all valuations on $\mathcal{A}$, $a \in A$ and formulas ϕ.

(b) Show that for each model $\mathcal{A}$ of TEMP the three conditions:

(i) For each formula ϕ, $\mathcal{A} \Vdash^u [\approx]\phi \to [-][+]\phi$.

(ii) For each formula ϕ, $\mathcal{A} \Vdash^u \langle + \rangle [\approx]\phi \to [+]\phi$.

(iii) For each wedge

$$\begin{array}{ccc} a & \xrightarrow{+} & b \\ {\scriptstyle +}\downarrow & & \\ c & & \end{array}$$

of elements we have $b \approx c$.

are equivalent.

(c) For an arbitrary model $\mathcal{A}$ of LINTIM:

(i) Show that the relation $\approx$ is the least equivalence relation which includes $\xrightarrow{+}$ (and $\xrightarrow{-}$).

(ii) Show that each $\approx$-equivalence class of $\mathcal{A}$ is linearly ordered by $\xrightarrow{+}$.

(d) Conversely, show that each disjoint union of linearly ordered sets provides a model for LINTIM.

8.13 This exercise continues Exercise 8.12. Each linearly ordered set $(A, <)$ produces a model of LINTIM (by interpreting $a \xrightarrow{+} b$ as $a < b$, etc). In particular

$$\mathcal{N} = (\mathsf{N}, <) \quad , \quad \mathcal{Z} = (\mathsf{Z}, <) \quad , \quad \mathcal{Q} = (\mathsf{Q}, <) \quad , \quad \mathcal{R} = (\mathsf{R}, <)$$

are models of LINTIM.

(a) Find a sentence ϕ modelled by each of $\mathcal{N}, \mathcal{Z}, \mathcal{Q}$, and $\mathcal{R}$ but for which $\neg[\vdash_{\mathsf{LINTIM}} \phi]$.

(b) Find sentences θ and ψ with

$$\mathcal{N} \Vdash^u \theta \quad , \quad \neg[\mathcal{Z} \Vdash^u \theta] \quad , \quad \neg[\mathcal{N} \Vdash^u \psi] \quad , \quad \mathcal{Z} \Vdash^u \psi .$$

(c) Find a formula shape which is modelled by both $\mathcal{Q}$ and $\mathcal{R}$ but by neither $\mathcal{N}$ nor $\mathcal{Z}$.

(d) Can you find a formula ϕ such that $\mathcal{R} \Vdash^u \phi$ but $\neg[\mathcal{Q} \Vdash^u \phi]$?

8.14 For some structures the interpretation of the two operators $\Box$ and $\Diamond$ agree, and in this case we often write $\bigcirc$ for both. Such structures are controlled by a function next and we think of this as a ticking clock.

(a) For a structure $\mathcal{A}$ with a transition relation $\longrightarrow$ corresponding to the modal operators $\Box$ and $\Diamond$, show that the following are equivalent.

(i) There is a function next : $A \longrightarrow A$ such that

$$a \longrightarrow b \Leftrightarrow \mathsf{next}(a) = b$$

holds for all $a, b \in A$.

(ii) For all formulas ϕ, $\mathcal{A} \Vdash^u \Box\phi \leftrightarrow \Diamond\phi$.

(b) Let $\mathbb{A}$ be the class of structures $\mathcal{A} = (A, \longrightarrow, \stackrel{\bullet}{\longrightarrow})$ where $\longrightarrow$ is a ticking clock (given by the function next) and $\stackrel{\bullet}{\longrightarrow}$ is the $*$-closure of $\longrightarrow$. Show that $\mathbb{A}$ models each of the shapes

$$\bigcirc\phi \leftrightarrow \neg\bigcirc\neg\phi \qquad [\bullet]\phi \leftrightarrow [\bullet]^2\phi$$

$$[\bullet]\phi \leftrightarrow \phi \wedge \bigcirc[\bullet]\phi \qquad [\bullet](\phi \rightarrow \bigcirc\phi) \rightarrow (\phi \rightarrow [\bullet]\phi)$$

(for all formulas ϕ).

8.15 Let S be any non-empty set and let Σ be any collection of non-empty subsets of S with $S \in \Sigma$. We call the elements s of S the *situations*, and we call the elements σ of Σ the *infons* (packets of information). We say a situation s *supports* an infon σ if $s \in \sigma$.

Let A be the set of supported infons i.e. the set of pairs

$$a = (s, \sigma) \quad \text{where} \quad s \in \sigma.$$

Define the relations $\stackrel{\approx}{\longrightarrow}$ and $\longrightarrow$ on A by

$$\begin{array}{lll} (s,\sigma) \stackrel{\approx}{\longrightarrow} (t,\tau) & \text{means} & \sigma = \tau \\ (s,\sigma) \longrightarrow (t,\tau) & \text{means} & \sigma = \tau \text{ and } t \in \tau \end{array}$$

and set

$$\mathcal{A} = (A, \stackrel{\approx}{\longrightarrow}, \longrightarrow).$$

(a) Show that $\stackrel{\approx}{\longrightarrow}$ is an equivalence relation, and $\longrightarrow$ is a partial ordering of A.

(b) Show that $\mathcal{A}$ models SL.

(c) Suppose that $S \in \Sigma$. Show that $\mathcal{A}$ models the confluence shape

$$\Diamond\Box\phi \rightarrow \Box\Diamond\phi$$

if and only if for each $\sigma, \tau \in \Sigma$ and $s \in \sigma \cap \tau$, there is some $\rho \in \Sigma$ with $s \in \rho \subseteq \sigma \cap \tau$.

8.16 As in Exercise 7.2 of Chapter 7, for each set Φ of formulas let Φ^* be the closure of Φ under $[i]$ for arbitrary labels i. Thus each member of Φ^* has the form

$$[\mathsf{i}]\theta$$

for some compound label i and $\theta \in \Phi$. Let S be an arbitrary standard formal system.

(a) Show that

$$\Phi \vdash_{\mathsf{S}} \Phi^*$$

and all three of the implications

$$\Phi \vdash^w_{\mathsf{S}} \phi \Rightarrow \Phi^* \vdash^w_{\mathsf{S}} \phi \Rightarrow \Phi \vdash_{\mathsf{S}} \phi \Rightarrow \Phi^* \vdash_{\mathsf{S}} \phi$$

hold for all formulas ϕ. Furthermore, show that the left hand implication is not reversible.

(b) Mimic the proof of the Deduction Theorem for propositional logic to show that the central and right hand implications of (a) are reversible.

(c) Extend the result of Exercise 7.2 to show that the three conditions

$$\Phi^* \models^p_{\mathsf{S}} \phi \quad , \quad \Phi^* \models^v_{\mathsf{S}} \phi \quad , \quad \Phi \models^v_{\mathsf{S}} \phi \quad ,$$

are equivalent.