Chapter 9

The general completeness result

9.1 Introduction

Let S be a standard formal system and let $\mathbb{M}$ be a class of structures suitable for the language of S. The members of $\mathbb{M}$ may be unadorned, valued, or valued and pointed as the case may be, but all must be of the same kind. We say that S and $\mathbb{M}$ are *completely matched* if for each formula ϕ we have

$$\vdash_{\mathsf{S}} \phi \quad \Leftrightarrow \quad \mathbb{M} \Vdash^{k} \phi$$

where $\Vdash^k$ is the satisfaction relation appropriate for the kind of $\mathbb{M}$. Such an equivalence is called a *completeness result*, where the implication $\Rightarrow$ is the *soundness* component and the implication $\Leftarrow$ is the *adequacy* component.

A completeness result of this kind is usually the solution to one of two different kinds of problems.

(a) Here we are given the system S and the problem is to find a class $\mathbb{M}$ which completely matches S. The reason for doing this is to analyse the properties of S in a more algebraic way, and the choice of $\mathbb{M}$ should take this into account.

(b) Here we are given $\mathbb{M}$ and the problem is to find a system S which completely matches $\mathbb{M}$. The reason in doing this is to obtain a uniform way of describing the common properties of the structures in $\mathbb{M}$. Thus it is desirable to make S as simple as possible.

Of course, for an arbitrarily given S or $\mathbb{M}$ there may be no matching partner, or there may be several. Thus proving a completeness result is not just a routine exercise. Nevertheless, such results exhibit some general features and there are some commonly used techniques. These will be described in this and the following chapters.

In this chapter we will prove a completeness result which is applicable to all standard systems S (and provides a solution to a problem of type (a)). This universal applicability means that the result is, in fact, rather weak; however the result does provide a basis for the vast majority of completeness results, in the sense that many of the proofs of these results are refinements of the universal proof.

9.2 Statement of the result

For the remainder of this chapter let S be a fixed, but arbitrary, standard formal system with axiom $\mathcal{S}$. We will produce a class of valued structures which completely matches S. In fact we will produce two such classes which, in some sense, are at opposite extremes.

For one extreme let $\mathbb{M}$ be the class of all valued structures which model S (i.e. are models of $\mathcal{S}$). Note that for each formula ϕ we have

$$\mathbb{M} \Vdash^v \phi \quad \Leftrightarrow \quad \models^v_S \phi$$

(for the right hand side is defined to mean the left hand side). Also, by the general soundness result, we have

$$\vdash_S \phi \quad \Rightarrow \quad \models^v_S \phi$$

so a completeness result will follow if we can prove the converse of this last implication.

For the other extreme we will construct a particular valued structure $(\mathfrak{S}, \sigma)$ – called the *canonical valued structure of* S – which models S (i.e. is a member of $\mathbb{M}$) and on its own completely matches S.

Putting these together we see that the eventual aim of this chapter is to prove the following completeness and characterization result.

9.1 THEOREM. *Let* S *be a standard formal system with canonical valued structure* $(\mathfrak{S}, \sigma)$. *For each formula* ϕ *the conditions*

(ii) $(\mathfrak{S}, \sigma) \Vdash^v \phi$
(iii) $\vdash_S \phi$
(iv) $\models^v_S \phi$

are equivalent.

(The numbering of the items of this result has been done to facilitate a comparison with a later result.)

Note that the implication (iii) $\Rightarrow$ (iv) is just soundness (which we have already proved in Chapter 8). The implication (iv) $\Rightarrow$ (ii) follows immediately we have verified that $(\mathfrak{S}, \sigma)$ models S. Much of the content of this theorem is in the implication (ii) $\Rightarrow$ (iii).

9.3 Maximally consistent sets

Intuitively a set of formulas Φ is inconsistent relative to S if the machinery of S can be used to derive a contradiction from Φ. Because of the failure of the Deduction Property we need to take a little care in making this idea precise. It turns out that the crucial property is the existence or not of formulas $\phi_1, \ldots, \phi_n$ in Φ such that

$$\vdash_S \phi_1 \wedge \ldots \wedge \phi_n \to \bot \tag{9.1}$$

The existence of such formulas means that Φ is inconsistent (relative to S), and the non-existence means that Φ is consistent. This can be made precise using the weak proof consequence relation $\vdash^w_S$.

9.2 DEFINITION.

(a) A set of formulas Φ is S-*consistent* if $\neg[\Phi \vdash^w_S \bot]$ i.e. if there are no members $\phi_1, \ldots, \phi_n$ of Φ for which (9.1) holds. Let $\boldsymbol{CON}$ be the collection of all such S-consistent sets.

(b) A set of formulas is *maximally* S-*consistent* if it is S-consistent but no proper extension of it is. Let $\boldsymbol{S}$ be the set of all such maximally S-consistent sets. ■

In more detail, each member s of $\boldsymbol{S}$ is a member of $\boldsymbol{CON}$ and for each formula ϕ

$$s \cup \{\phi\} \in \boldsymbol{CON} \quad \Rightarrow \quad \phi \in s$$

holds (for if $\phi \notin s$ then $s \cup \{\phi\}$ is a proper extension of s and so cannot be consistent). This maximality ensures that each $s \in \boldsymbol{S}$ has several useful closure properties. Clearly, there is no formula ϕ such that both ϕ and $\neg\phi$ are in s (for, trivially,

$$\vdash_S \sigma \wedge \neg\phi \to \bot).$$

Also, if $\phi \notin s$ then $s \cup \{\phi\}$ is not consistent, so there is a conjunction σ of finitely many member of s with

$$\vdash_S \sigma \wedge \phi \to \bot. \tag{9.2}$$

Similarly, if $\neg\phi \notin s$ then

$$\vdash_S \tau \wedge \neg\phi \to \bot$$

for some conjunction τ of members of s. But then, using an appropriate tautology, we have

$$\vdash_S \sigma \wedge \tau \to \bot$$

which would mean that s is, in fact, inconsistent. Since this is not so, at least one of ϕ and $\neg\phi$ is in s, and hence s contains precisely one of ϕ and $\neg\phi$.

A similar argument shows that s is closed under implication. For suppose, for some formula ϕ, there is a conjunction ρ of members of s with

$$\vdash_{\mathsf{S}} \rho \to \phi.$$

Then ϕ is in s, for otherwise there is a suitable conjunction σ such that (9.2) holds and hence

$$\vdash_{\mathsf{S}} \sigma \wedge \rho \to \bot$$

which would mean that s is inconsistent.

By continuing in this way we may arrive at a proof of the following Proposition. (The remaining details are left as an exercise.)

9.3 PROPOSITION. *Let $s \in \boldsymbol{S}$. Then*

$$\top \in s \quad , \quad \bot \notin s$$

and for all formulas θ, ψ, ϕ, the equivalences

$$\begin{array}{llcl} (\neg) & \neg\phi \in s & \Leftrightarrow & \phi \notin s \\ (\wedge) & \theta \wedge \psi \in s & \Leftrightarrow & \theta \in s \ \ \textit{and} \ \ \psi \in s \\ (\vee) & \theta \vee \psi \in s & \Leftrightarrow & \theta \in s \ \ \textit{or} \ \ \psi \in s \\ (\to) & \theta \to \psi \in s & \Leftrightarrow & \theta \notin s \ \ \textit{or} \ \ \psi \in s \end{array}$$

hold.

Note that this Proposition does not contain a clause corresponding to the connective $[i]$. The appropriate property for this will be dealt with in the next section.

The most important property of $\boldsymbol{S}$ is that it is non-empty and, in fact, has enough members to distinguish between all formulas which ought to be distinguishable. The precise result is as follows.

9.4 LEMMA. (Basic Existence Result) *For each S-consistent set of formulas Φ there is some $s \in \boldsymbol{S}$ with $\Phi \subseteq s$.*

Proof. Let $(\phi_r \mid r < \omega)$ be any enumeration of all formulas, and define the sequence $(\Delta_r \mid r < \omega)$ of sets of formulas by

$$\begin{array}{lcl} \Delta_0 & = & \Phi \\ \Delta_{r+1} & = & \begin{cases} \Delta_r \cup \{\phi_r\} & \text{if this is S-consistent} \\ \Delta_r & \text{otherwise.} \end{cases} \end{array}$$

By construction we have

$$\Phi = \Delta_0 \subseteq \Delta_1 \subseteq \cdots \subseteq \Delta_r \subseteq \cdots \qquad (r < \omega)$$

and each Δ_r is S-consistent. Let

$$s = \bigcup\{\Delta_r \mid r < \omega\}.$$

This set s is consistent. For if it isn't, then some finite part of s is inconsistent, and this finite part is included in some Δ_r, which is consistent. Thus it remains to show that s is maximal.

Consider any formula ϕ such that $s \cup \{\phi\} \in \boldsymbol{CON}$. There is at least one index $r < \omega$ such that $\phi = \phi_r$. But then

$$\Delta_r \cup \{\phi_r\} \subseteq s \cup \{\phi\}$$

so that the smaller set is also consistent, and hence

$$\phi = \phi_r \in \Delta_{r+1} \subseteq s$$

to demonstrate the required maximality of s. ■

As remarked already, this is the most important result of the whole chapter. In fact Theorem 9.1 (as is also the case with several other results) is little more than a rewording of Lemma 9.4. Almost all of what follows is a slightly elaborate exercise in symbol shuffling.

A simple consequence of the Basic Existence Result (i.e. of Lemma 9.4) is a characterization of S-derivability from a set of hypotheses.

9.5 LEMMA. *For each set of formulas Φ and formula ϕ, the equivalence*

$$\Phi \vdash^{w}_{\mathsf{S}} \phi \quad \Leftrightarrow \quad (\forall s \in \boldsymbol{S})[\Phi \subseteq s \Rightarrow \phi \in s]$$

holds.

Proof. ($\Rightarrow$). This holds since each $s \in \boldsymbol{S}$ is closed under implication.

($\Leftarrow$). The hypothesis (ii) together with the Basic Existence Result ensure that

$$\Phi \cup \{\neg\phi\} \notin \boldsymbol{CON}.$$

But then there is a conjunction τ of members of Φ such that

$$\vdash_{\mathsf{S}} \tau \wedge \neg\phi \to \bot.$$

An application of a tautology now gives (i). ■

One particular case of this Lemma is worth noting separately, namely the case $\Phi = \emptyset$.

9.6 COROLLARY. *For each formula ϕ, the equivalence*

$$\vdash_{\mathsf{S}} \phi \quad \Leftrightarrow \quad (\forall s \in \boldsymbol{S})[\phi \in s]$$

holds.

9.4 The canonical structure

Each structure $\mathcal{A}$ (labelled transition structure) comprises a non-empty carrying set A furnished with an appropriately indexed family of binary relations (transition relations) $\xrightarrow{i}$ (one for each label i). The canonical structure $\mathfrak{S}$ for S is such a structure based on the set $\boldsymbol{S}$ of maximally consistent sets of formulas. To complete the construction of $\boldsymbol{S}$ it remains to define the corresponding family of relations. Thus, for each index i let

$$\xrightarrow{i}$$

be the relation on $\boldsymbol{S}$ where for each $s, t \in \boldsymbol{S}$,

$$s \xrightarrow{i} t$$

holds precisely when for all formulas ϕ,

$$[i]\phi \in s \quad \Rightarrow \quad \phi \in t.$$

We also use

$$t \prec_i s$$

to indicate that $s \xrightarrow{i} t$ holds. (As usual, this will help to condense certain descriptions.)

Note how the $(\neg)$ clause of Proposition 9.3 shows that $s \xrightarrow{i} t$ holds exactly when for each formula ϕ

$$\phi \in t \quad \Rightarrow \quad \langle i \rangle \phi \in s.$$

The Basic Existence Result now allows us to add to the equivalences of this Proposition.

9.7 LEMMA. *For each $s \in \boldsymbol{S}$ and formula ϕ, the equivalence*

$$[i]\phi \in s \quad \Leftrightarrow \quad (\forall t \prec_i s)[\phi \in t]$$

holds.

Proof. $(\Rightarrow)$ This follows immediately from the definition of $\prec_i$.
$(\Leftarrow)$ Consider set of formulas

$$\Psi = \{\psi \mid [i]\psi \in s\}.$$

Then the definition of $\prec_i$ and the hypothesis (the right hand side) gives

$$\Psi \subseteq t \in \boldsymbol{S} \quad \Rightarrow \quad t \prec_i s \quad \Rightarrow \quad \phi \in t$$

so that Lemma 9.5 provides $\psi_1, \ldots, \psi_n \in \Psi$ with

$$\vdash_{\mathsf{S}} \psi_1 \wedge \ldots \wedge \psi_n \rightarrow \phi.$$

Using the basic properties of S, this gives

$$\vdash_{\mathsf{S}} [i]\psi_1 \wedge \ldots \wedge [i]\psi_i \rightarrow [i]\phi.$$

Since each $[i]\psi_r \in s$ and s is closed under implications, this gives $[i]\psi \in s$, as required. ■

9.5 The canonical valuation

As we will see later, in general the unadorned canonical structure $\mathfrak{S}$ is not a model of S. However, it does model S when enriched by suitably chosen valuations. The *canonical valuation* σ on $\mathfrak{S}$ is given by

$$\sigma(P) = \{s \in \boldsymbol{S} \mid P \in s\}$$

for variables P. Equivalently, σ is such that

$$s \Vdash P \quad \Leftrightarrow \quad P \in s$$

for $s \in \boldsymbol{S}$ and variables P. This equivalence extends naturally.

9.8 LEMMA. *For each $s \in \boldsymbol{S}$ and formula ϕ, the equivalence*

$$s \Vdash \phi \quad \Leftrightarrow \quad \phi \in s$$

holds.

Proof. For each formula ϕ consider the condition

$$(\phi) \qquad (\forall s \in \boldsymbol{S})[s \Vdash \phi \Leftrightarrow \phi \in s].$$

We verify (ϕ) by induction on the complexity of ϕ.

The base case holds by the definition of σ (and since $\top \in s$ and $\bot \notin s$). The passage across the propositional connectives follow from the equivalences of Proposition 9.3. It thus remains to pass across $[i]$.

For a given $s \in \boldsymbol{S}$ and formula ϕ, using first the definition of $\Vdash$ and then the Induction Hypothesis (ϕ) followed by Lemma 9.7, we have

$$\begin{aligned} s \Vdash [i]\phi \quad &\Leftrightarrow \quad (\forall t \prec_i s)[t \Vdash \phi] \\ &\Leftrightarrow \quad (\forall t \prec_i s)[\phi \in t] \quad \Leftrightarrow \quad [i]\phi \in s \end{aligned}$$

which gives $([i]\phi)$, and so completes the proof. ■

As an immediate consequence of this with Corollary 9.6 we have the following.

9.9 COROLLARY. *The canonical valued structure $(\mathfrak{S}, \sigma)$ models* S.

9.6 Proof of the result

The proof of Theorem 9.1 is now very short. For instance, for each formula ϕ, Lemma 9.8 and Corollary 9.6 give

$$\begin{aligned}(\mathfrak{S},\sigma) \Vdash^v \phi &\Leftrightarrow (\forall s \in \boldsymbol{S})[s \Vdash \phi] \\ &\Leftrightarrow (\forall s \in \boldsymbol{S})[\phi \in s] \quad \Leftrightarrow \quad \vdash_{\mathsf{S}} \phi\end{aligned}$$

which verifies the implication (ii) $\Rightarrow$ (iii). The implication (iii) $\Rightarrow$ (iv) is just soundness, and the implication (iv) $\Rightarrow$ (ii) is a consequence of Corollary 9.9. ■

9.7 Concluding remarks

On the face of it, Theorem 9.1 gives us two solutions to problem (a) (of Section 9.1) for an arbitrary standard system S. However, these solutions are not of much practical value. On the one hand we are told we may analyse S by looking at the class of all its valued models. Unfortunately, to investigate this large class we must already have a fairly extensive knowledge of S (in which case we do not need the completeness result). On the other hand we are told we may analyse S by looking at a particular valued model. Unfortunately, this model is constructed from S and the process of determining its properties involves an analysis of S (which makes the exercise somewhat pointless).

In spite of these drawbacks, Theorem 9.1 does have some value. It does at least tell us that every standard system has a single characteristic valued model (a fact which is not at all obvious). However, to bring out its full potential, we must now refine the Theorem. We may do this by either strengthening the conclusion, or by modifying the proof to extract more information.

There are two possible lines of development.

One possibility is to try to eliminate the references to valuations, and look for a characteristic class $\mathbb{M}$ consisting of unadorned structures. This can be done for certain pleasantly disposed systems S (for which $\mathfrak{S}$ itself models S). It turns out that such systems S have a characteristic class $\mathbb{M}$ defined entirely without reference to S, and hence the completeness result does open up a genuine second line of attack on S. This case is discussed in the next chapter.

Another possibility is to look for a characteristic class which consists entirely of finite structures. This then opens up the possibility of a mechanical test for derivability within the system. (For, clearly, checking validity in a finite structure is potentially mechanizable.) Again we find there are many systems for which this approach is feasible, and these are discussed in a later chapter.

There are, of course, systems with no known completeness result (beyond that of Theorem 9.1) and there are examples of theories with various demonstrable complexities (or eccentricities). A first course in modal logic is not the

place for a detailed account of these, but such examples should at least be mentioned. (Otherwise you might get an over rosy view of modal life.) A few of these more complex systems will be described later.

9.8 Exercises

9.1 Fill in the details of the proof of Proposition 9.3.

9.2 This exercise continues and makes use of the notation of and is related to the content of Exercise 7.2 and 8.16. Thus, for an arbitrary formal system S and set Φ of formulas, let $\boldsymbol{S}(\Phi)$ be the set of all $s \in \boldsymbol{S}$ with $\Phi^* \subseteq s$. This set $\boldsymbol{S}(\Phi)$ is converted into a transition structure $\mathfrak{S}(\Phi)$ using the restriction to $\boldsymbol{S}(\Phi)$ of the transition relations of $\mathfrak{S}$. Thus $\mathfrak{S}$ is the particular case $\mathfrak{S}(\emptyset)$. In the same way let σ be the restriction to $\mathfrak{S}(\Phi)$ of the canonical valuation on $\mathfrak{S}$.

(a) Show that for each $s \in \boldsymbol{S}(\Phi)$, formula ϕ, and label i, the conditions

- $[i]\phi \in s$
- for each $t \in \boldsymbol{S}(\Phi)$, if $s \xrightarrow{i} t$ then $\phi \in t$

are equivalent.

(b) Show that

$$s \Vdash \phi \Leftrightarrow \phi \in s$$

holds for all $s \in \boldsymbol{S}(\Phi)$ and formulas ϕ.

(c) Show that $(\mathfrak{S}(\Phi), \sigma)$ models $\mathsf{S} \cup \Phi^*$.

(d) Show that for each formula ϕ, the conditions

(i) $\Phi^* \vdash^w_{\mathsf{S}} \phi$

(ii l) $\Phi \vdash_{\mathsf{S}} \phi$ (ii r) $\Phi \models^v_{\mathsf{S}} \phi$

(iii l) $\Phi^* \vdash_{\mathsf{S}} \phi$ (iii r) $\Phi^* \models^v_{\mathsf{S}} \phi$

(iv) $(\mathfrak{S}(\Phi), \sigma)$ models ϕ

are equivalent.