Hybrid Logic Tango

University of Buenos Aries

11–15 February 2008

Exam

Due by: **March 31st, 2008**

Answer three questions.

Question 1: In Lecture 1 we gave the following important definition:

> Let $\mathcal{M} = (W, \mathcal{R}, V)$ and $\mathcal{M}' = (W', \mathcal{R}', V')$ be models for the same basic hybrid language. A relation $Z \subseteq W \times W'$ is a *bisimulation-with-constants* between $\mathcal{M}$ and $\mathcal{M}'$ if the following conditions are met:
>
> 1. Atomic harmony: if wZw' then $w \in V(p)$ iff $w' \in V'(p)$, for all propositional symbols p, and all nominals i.
> 2. Forth: if wZw' and wRv then there is a v' such that $w'R'v'$ and vZv'.
> 3. Back: if wZw' and $w'R'v'$ then there is a v such that wRv and vZv'.
> 4. All points named by nominals are related by Z.

Prove that basic hybrid formulas are invariant under bisimulations-with-constants. That is, show the following:

> Let $\mathcal{M} = (W, \mathcal{R}, V)$ and $\mathcal{M}' = (W', \mathcal{R}', V')$ be models for the same basic hybrid language, and let Z be a bisimulation-with-constants between $\mathcal{M}$ and $\mathcal{M}'$. Then for all basic hybrid formulas φ, and all points w in $\mathcal{M}$ and w' in $\mathcal{M}$ such that w is bisimilar to w':
>
> $$\mathcal{M}, w \models \varphi \text{ iff } \mathcal{M}', w' \models \varphi.$$

Question 2: Let us say that a model (W, R, V) is *named* if every state in the model is the denotation of some nominal (that is, for all $w \in W$ there is some nominal i such that $V(i) = \{w\}$). Furthermore, if ϕ is a pure formula, we say that ψ is a *pure instance* of ϕ if ψ is obtained from ϕ by uniformly substituting nominals for nominals. Then the *Magic Lemma* says that if all pure instances of a pure formula ϕ are true at all points of a named model, then ϕ is valid on the frame underlying the model. That is:

Let $\mathcal{M} = (\mathfrak{F}, V)$ be a named model and ϕ a pure formula. Suppose that for all pure instances ψ of ϕ, all $w \in W$, $\mathcal{M}, w \models \psi$. Then $\mathfrak{F} \models \phi$.

Prove the Magic Lemma.

Question 3: For any $n \geq 1$, let $R^n xy$ be the first-order formula

$$\exists z_1 \cdots \exists z_n\, (Rxz_1 \wedge Rz_1z_2 \wedge \cdots \wedge Rz_ny).$$

Let ψ be a first-order formula that is a boolean combination of formulas of the form $R^n xy$, Rxy, and $x = y$. Show that the class of frames defined by the universal closure of ψ is definable in the basic hybrid language.

Hint: the trichotomy condition, which we mentioned in class is expressed in first-order logic by

$$\forall x \forall y (Rxy \vee x = y \vee Ryx)$$

and we defined this in hybrid logic by $@_j \Diamond i \vee @_j i \vee @_i \Diamond j$.

Question 4: In Lecture 5 we discussed the *universal modality* (or the *global modality*, as it is sometimes called). Recall that the box form of this modality is written A and the diamond form is written E. They have the following semantics. For any model $\mathcal{M} = (W, R, V)$, and any $w \in W$:

$$\begin{array}{lll} \mathcal{M}, w \models \mathrm{A}\phi & \text{iff} & \text{for all } u \in W, \mathcal{M}, u \models \phi \\ \mathcal{M}, w \models \mathrm{E}\phi & \text{iff} & \text{for some } u \in W, \mathcal{M}, u \models \phi \end{array}$$

Add the universal E to the basic hybrid language. Use a filtration argument to show that the satisfiability problem for the resulting language is decidable. (Recall that $@_i\phi$ can be defined to be $\mathrm{E}(i \wedge \phi)$ or $\mathrm{A}(i \rightarrow \phi)$, so you do not have to deal explicitly with the satisfaction operators.)

Question 5: In Lecture 3 we proved a completeness result for the basic hybrid language with respect to named models. Now, recall that the axioms we used made heavy use of the @-operator, and so did the PASTE rule, which played a crucial role in the proof. Now, suppose we wanted to use nominals, but we didn't want to use the @-operator. That is, supposed we wanted to drop @ from the basic hybrid language. Could we still prove completeness with respect to named model?

Yes, we can, and that's the purpose of the present exercise. Using the hints given below, write down such an axiomatisation, and prove it sound and complete with respect to named models.

First hint. Note that for all $n, m \geq 0$, the formula

$$\Diamond^n (i \wedge p) \rightarrow \Box^m (i \rightarrow p).$$

is valid. Here, for all $n, m \geq 0$, $\Diamond^n$, and $\Box^m$ are sequences of diamonds and boxes of lengths n and m respectively. Why are these axioms important? Essentially they let us prove that nominals really do occur uniquely in one MCS in any generated submodel of the canonical model. Think about it.

Second hint. Let $\Diamond_i \phi$ be a shorthand for $\Diamond(i \wedge \phi)$, and add all rules of the form

$$\frac{\models \Diamond_k \cdots \Diamond_i \Diamond_j \phi \to \theta}{\models \Diamond_k \cdots \Diamond_i \Diamond \phi \to \theta} \ .$$

Here j is a nominal distinct from $k, \cdots, i$ that does not occur in ϕ or θ.) Why are these rules important? Essentially they let us do what the PASTE rule does without using the @ operator.

Third hint. Note that the NAME RULE does not use the @ operator so we are still free to use it.

This is a harder exercise than the previous ones (we'll give you a star if you chose to solve it), but you will learn a lot about hybrid completeness proofs if you do it.