Hybrid Logic Tango
University of Buenos Aires
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Lecture 2: 12 February 2008

Lecture 2: The downarrow binder

Another name for this lecture would be "Taking Slogan 2 seriously":

Slogan 2: Modal languages provide an internal, local perspective on relational structures.

This is because we are going to discuss $\downarrow$, the downarrow binder. This lets us dynamically name the here-and-now. As we shall see, this seemingly-simple device cooperates with @ to capture the first-order logic of locality. We'll:

- Motivate $\downarrow$ via examples.
- Define its syntax, semantics, standard translation, and tableau rules.
- Mention some of its more important properties, most notably the fact that it captures the first-order logic of locality.

A home for modal logic

Claim: if you're doing traditional modal logic, you're working in the space carved out by hybrid logic with downarrow.

- We identify "locality" with "invariance under generated submodels."
- All traditional modal logics enjoy this property (though some newcomers, such as the global modality and the difference operator, explore what happens when you break locality).
- Hybrid logic with downarrow provides a comfortable home for traditional logics, performing such services as interpolation repair.

But we are getting *way* ahead of ourselves — let's first sit back and learn what downarrow actually does...

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But none of these formulas pins down the concept of self-respect — only the concepts of self-respect for i, for j, for k, and so on. We need to abstract away from the effects of particular nominals (constants).

Losers via downarrow

With the aid of the downarrow operator, we can do precisely this:

$$\downarrow x.\neg\langle \text{RESPECT}\rangle\, x$$

This says: Let x be a temporary name for the point in the model at which the formula is being evaluated. Then x is not related to x by the RESPECT relation.

To put it another way, it says: this person x (whoever that is) does not respect himself/herself. In a sense, its what linguists call a deictic definition.

The formula is true of precisely those states of our models (people) who do not respect themselves, so we have defined the required concept.

Example 2: Locally reflected epistemic states

In our second example, we'll think of the states of our models as epistemic states, and the relation between states as meaning is an epistemic alternative to (so we're in a traditional agent-based setting).

Let's say that an epistemic state s is locally reflected if all epistemic alternatives t to s have s as an epistemic alternative.

More precisely, s is locally reflected iff $\forall t(sRt \rightarrow tRs)$. That is, s is a locally reflected state if it is symmetrically linked to other points in the model.

Is there a basic hybrid formula that (in any model) distinguishes locally reflected from non locally reflected states?

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- On the other hand, if we insist that $@_i\Box\Diamond i$ is to be taken as a validity (that is, as an axiom) then we distinguish symmetric models from all other models. But that's not what we want either.
- We want a formula that classifies the states of a model into locally reflected and non locally reflected states.

Locally reflected states via downarrow

We can do this with downarrow. Instead of $@_i \Box \Diamond i$ we use:

$$\downarrow x.\Box \Diamond x$$

Paraphrase this as follows: "this epistemic state x (whichever it might be) is such that all it's epistemic alternatives have x as an epistemic alternative."

Again, we're defining the required concept by some kind of deixis (this time, deictic reference to epistemic states, not people).

Technically, we bind a state variable x to the current state. (A state variable is just like a nominal, except that it can be bound, whereas ordinary nominals can't.)

Example 3: Problems and alarms

In this example we'll think of the states of our models as points of time (so we're in a temporal logic setting). The example is adapted from "Temporal Logic with Forgettable Past", Laroussinie, Markey, and Schnoebelen, 17th IEEE Symp. Logic in Computer Science (LICS 2002), Copenhagen, Denmark, July 2002.

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- Easy: $[\text{F}]\,(\text{alarm} \rightarrow \langle \text{P} \rangle\, \text{problem})$ specifies this. We don't even need hybrid logic.

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- But this is probably not what we want — a problem that occurred before the reset may account for the alarm going off.

Resetting the alarm with downarrow

But with the aid of $\downarrow$ we can specify what we want. We dynamically name the spot where the reset occurred by binding the state variable x to it, and then demand that the problem occurred later than this:

$$[\mathrm{F}]\,(\mathrm{reset} \rightarrow \downarrow x.[\mathrm{F}]\,(\mathrm{alarm} \rightarrow \langle \mathrm{P} \rangle\,(\mathrm{problem} \wedge \langle \mathrm{P} \rangle\, x)))$$

Example 4: The *Until* operator

In this example we'll continue to think of the states of our models as points of time (so we're still doing temporal logic).

Hans Kamp's celebrated *Until* operator is a binary modality with the following satisfaction definition:

$$\mathcal{M}, w \Vdash \mathit{Until}(\varphi, \psi) \quad \text{iff} \quad \exists v(w < v \;\&\; \mathcal{M}, v \Vdash \varphi \;\&\; \forall u(w < u < v \Rightarrow \mathcal{M}, u \Vdash \psi))$$

This operator (and some of its variants) has proved extremely useful as a specification tool. Can we define it in basic hybrid logic?

Defining *Until* with downarrow

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No, we can't. But we can with the help of downarrow:

$$\mathit{Until}(\varphi, \psi) \;:=\; \downarrow x.\Diamond\downarrow y.(\varphi \wedge @_x\Box(\Diamond y \rightarrow \psi)).$$

This says: name the present state x. Then, by looking forward we can see a state (which we label y) such that φ is true at y, and every state between x and y verifies ψ.

Note the use of $@_x$ to jump back to x, the starting point. This is the first glimpse of a theme that echos through today's lecture: $\downarrow$ and @ work well together. We can use $\downarrow$ to 'store' some state of interest, and @ to 'retrieve' it when needed.

Syntax

- Here are the required syntactic changes. Choose a denumerably infinite set SVAR $= \{x, y, z \ldots\}$, the set of state variables, disjoint from PROP, NOM and MOD.
- Like nominals, state variables are atomic formulas which name states, but unlike nominals they can be bound.
- The hybrid language with downarrow (over PROP, NOM, MOD and SVAR) is defined as follows:

$$\begin{aligned} \text{WFF} \quad := \quad & x \mid i \mid \mathrm{p} \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \vee \psi \\ & \mid \varphi \rightarrow \psi \mid \langle \mathrm{M} \rangle \varphi \mid [\mathrm{M}] \varphi \mid @_i \varphi \mid @_x \varphi \mid \downarrow x.\varphi \end{aligned}$$

- Free and bound occurrences of state variables are defined in the expected way, with $\downarrow$ as the only binder. A sentence is a formula containing no free state variables.

Semantics

- Models $\mathcal{M}$ for hybrid languages with downarrow are just the hybrid models we are used to (as usual, nominals are assigned singletons).
- Given a model $\mathcal{M} = (W, R, V)$, an assignment on $\mathcal{M}$ is a function $g : \mathrm{SVAR} \longrightarrow W$. (Thus an assignment makes a state variable true at precisely one state.)
- Assignments will be used to interpret free state variables Tarski-style. We merely relativise the clauses of the satisfaction definition for the basic hybrid language to assignments, and add the three new clauses we require. Here's how ...

Satisfaction Definition

$$
\begin{array}{lll}
\mathcal{M}, g, w \Vdash x & \text{iff} & w = g(x) \text{ where } x \in \text{SVAR} \\
\mathcal{M}, g, w \Vdash @_x \varphi & \text{iff} & \mathcal{M}, g, g(x) \Vdash \varphi \\
\mathcal{M}, g, w \Vdash \varphi \wedge \psi & \text{iff} & \mathcal{M}, g, w \Vdash \varphi \text{ and } \mathcal{M}, g, w \Vdash \psi \\
\mathcal{M}, g, w \Vdash \Diamond \varphi & \text{iff} & \exists w' (wRw' \ \& \ \mathcal{M}, g, w' \Vdash \varphi) \\
\mathcal{M}, g, w \Vdash \downarrow x.\varphi & \text{iff} & \mathcal{M}, g', w \Vdash \varphi, \text{ where } g' \overset{x}{\sim} g \text{ and } g'(x) = w
\end{array}
$$

The fifth clause defines $\downarrow$ to be an operator that binds variables to the state w at which evaluation is being performed. The notation $g' \overset{x}{\sim} g$ means that g' is the assignment that differs from g, if at all, only in what it assigns to x. By stipulating that $g'(x)$ is to be w, we bind a label to the here-and-now.

For sentences φ, we can simply write $\mathcal{M}, w \Vdash \downarrow x.\varphi$ — why is this?

Standard Translation

Assume we're using the same symbols for both state variables and first-order variables. Let s be a metavariable over state variables and nominals.

$$\begin{aligned} \mathrm{ST}_x(y) &= (y = x) \\ \mathrm{ST}_x(@_s\varphi) &= \mathrm{ST}_s(\varphi) \\ \mathrm{ST}_x(\downarrow y.\varphi) &= \exists y(x = y \wedge \mathrm{ST}_x(\varphi)) \end{aligned}$$

This translation is satisfaction preserving, so hybrid logic with downarrow is a fragment of the correspondence language (with constants and equalities). We'll see later which fragment it corresponds to.

Tableau rules

We only need to make two changes. First, we need to let our previous tableau rules apply when the subscript on @ is a state variable rather than a nominal.

Second, we add the following two rules to cope with $\downarrow$. In the following rule, s is used as a metavariable over nominals and state variables:

$$\frac{@_s{\downarrow}x.\varphi}{@_s\varphi[x \leftarrow s]} \qquad \frac{\neg@_s{\downarrow}x.\varphi}{\neg@_s\varphi[x \leftarrow s]}$$

If s is a variable, before substituting we rename bound occurrences of s in φ to prevent accidental capture.

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4	$@_i\neg\varphi[x \leftarrow i]$	$\downarrow$ rule on 3

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6	$@_i\varphi[x \leftarrow i]$	$\downarrow$ rule on 2

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6	$@_i\varphi[x \leftarrow i]$	$\downarrow$ rule on 2
	$\bot_{5,6}$	

Completeness

This tableau system is (sound and) complete with respect to the class of all models.

Nonetheless, as was explained in yesterday's lecture, we are often interested in deduction over other classes of models. Can the tableau system be extended to deal with reasoning over other classes of models?

Yes, it can — and once again it's pure formulas that make things easy.

Pure formulas

- As before, a pure formula is simply a formula not containing any propositional symbols.
- But this means that pure formulas may contain state variables and $\downarrow$, not just nominals, $\bot$ and $\top$, so we can define a lot more frame classes than before.
- Nonetheless, completeness is still automatic. Recall that if $@_i\varphi$ be a pure formula, whose nominals (if any) are $i, i_1, \ldots, i_n$, then we can turn it into the following tableau rule:

$$\frac{(j, j_1, \ldots, j_n \text{ on branch})}{@_i\varphi[i \leftarrow j, i_1 \leftarrow j_1, \ldots, i_n \leftarrow j_n]}$$

Frame definability and deduction match for pure formulas

Completeness Theorem Suppose you extend the basic tableau system with the tableau rules for the pure formulas $@_j\varphi$, ..., $@_k\psi$ (that is, the rules of the form just described). Then the resulting system is (sound and) complete with respect to the class of frames defined by these formulas.

That is, the frame-defining and deductive powers of pure formulas match perfectly — even when $\downarrow$ has been added to the language.

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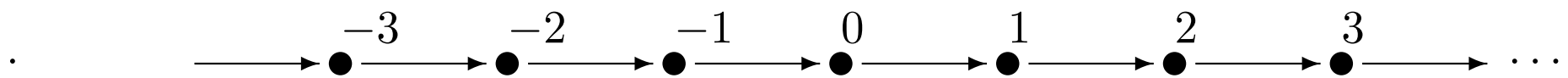
To understand what this means we're going to need to learn something about submodels and generated submodels...

Submodels

Suppose $\mathcal{M}$ is a model based on this frame (the integers in their usual order):



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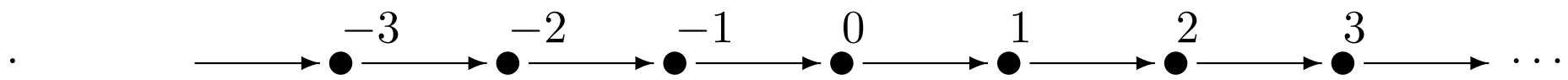
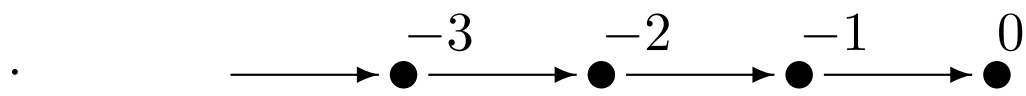


Suppose we form a submodel $\mathcal{M}^-$ of $\mathcal{M}$ by throwing away all the positive numbers, and restricting the original valuation (whatever it was) to the remaining numbers:



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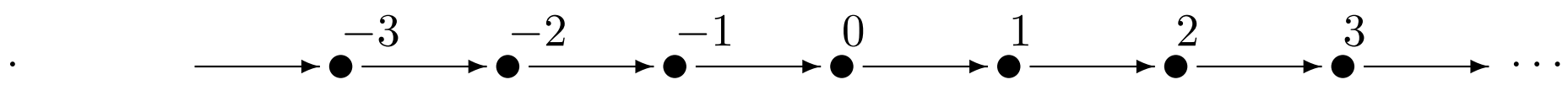

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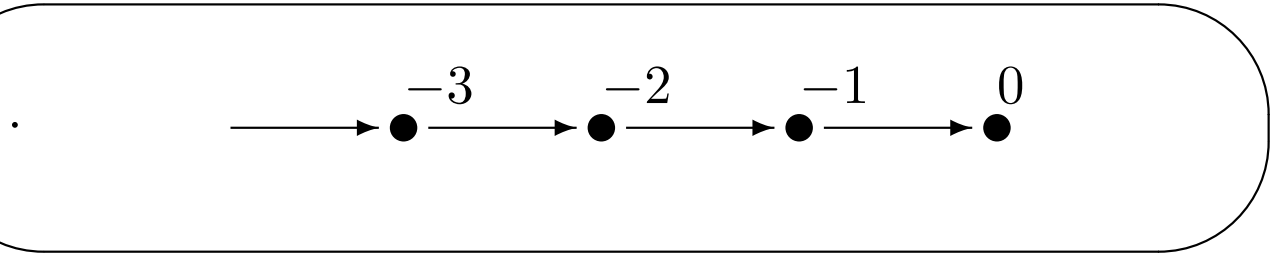

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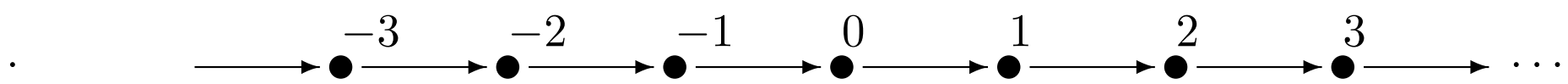

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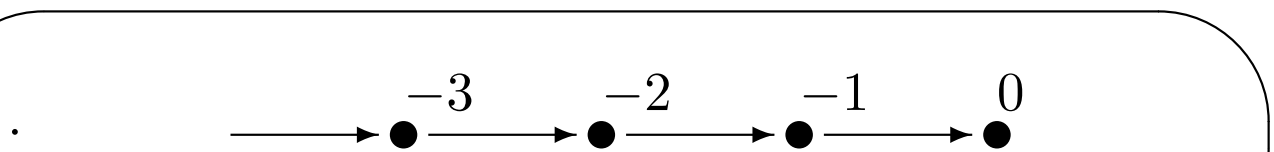

Can an orthodox modal language detect the difference between the two model?

Yes! $\mathcal{M}, 0 \Vdash \Diamond\top$, but $\mathcal{M}^-, 0 \nVdash \Diamond\top$

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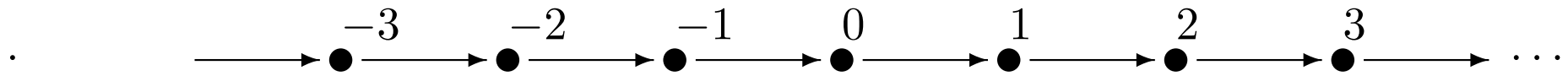

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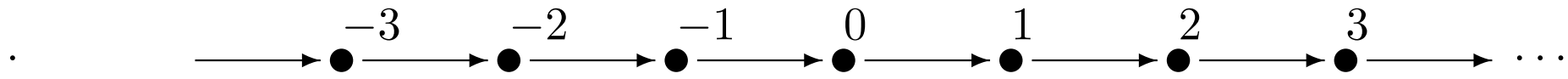
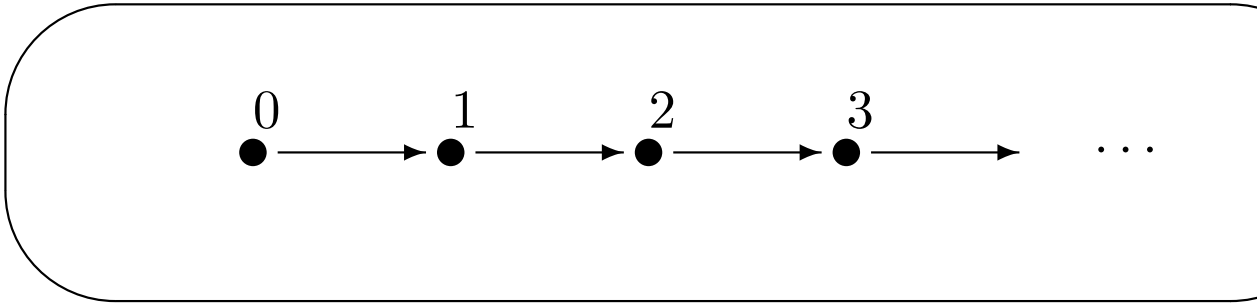


This time, suppose we form a submodel of $\mathcal{M}^+$ of $\mathcal{M}$ obtained by throwing away the negative numbers, and restricting the original valuation to what remains:



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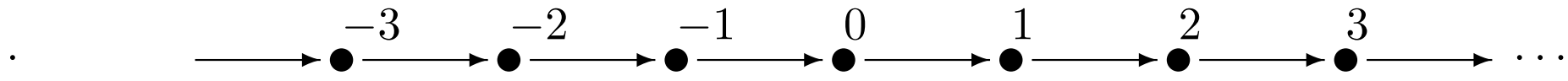

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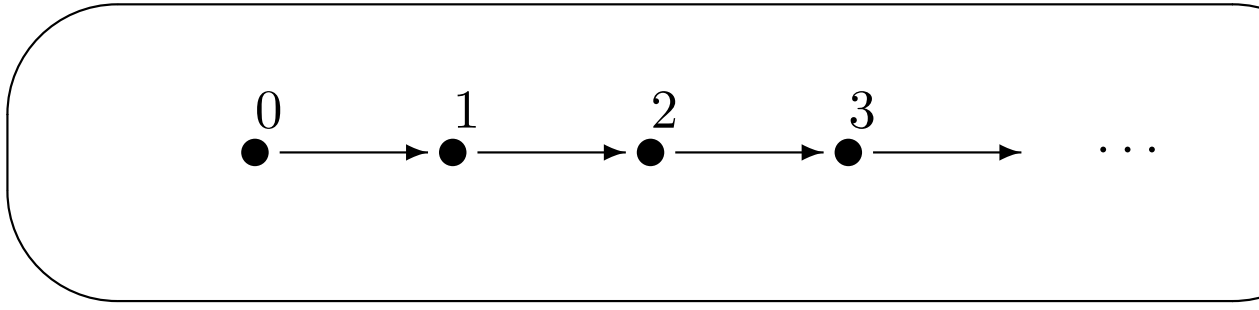

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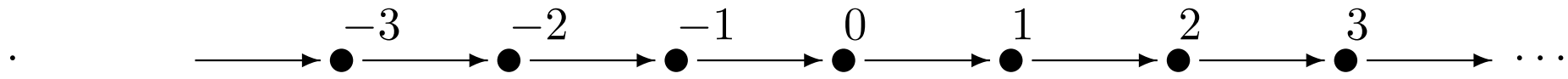

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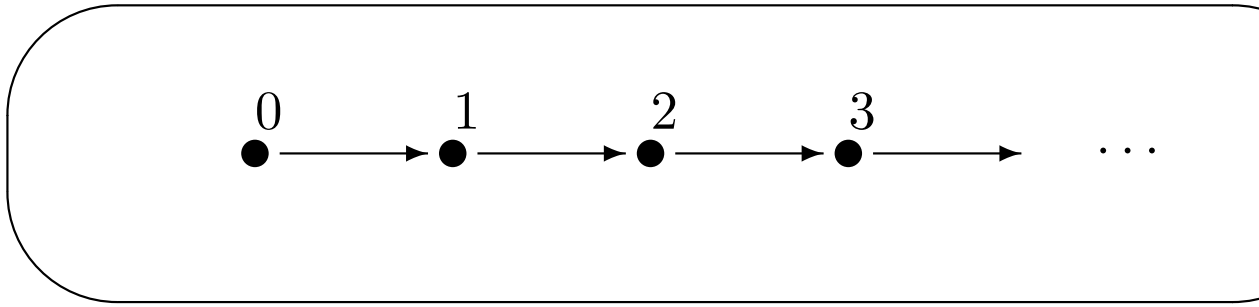

Can an orthodox modal language detect the difference between the two model?

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- Well, in the second example the two models were bisimilar, and in the first example they weren't.
- But there's a more direct intuition: the second model consisted of the point 0 and all it's successors (that is, it's the submodel generated by the point 0).
- To put it another way, point generation selects all the points that are reachable from the evaluation state by chaining through the relation(s). It selects precisely the points needed to satisfy a formula at some particular location, and ignores the rest.

Point generated submodels

Point generated submodels Let $\mathcal{M} = (W, R, V)$ be a model, and $w \in W$. Let $W_w = \{w' \in W \mid wR^*w'\}$, where R^* is the reflexive transitive closure of R. Then $\mathcal{M}_w$, the submodel of $\mathcal{M}$ generated by w is the model (W_w, R_w, V_w) where R_w and V_w are the restrictions of R and V, respectively, to W_w.

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Proposition: Let $\mathcal{M}$ be a model and $\mathcal{M}_w$ any of its point generated submodels. Then for any orthodox modal formula φ, and any point u in $\mathcal{M}_w$ we have

$$\mathcal{M}, u \Vdash \varphi \text{ iff } \mathcal{M}_w, u \Vdash \varphi$$

In words: model satisfaction is invariant for point generated submodels.

Proof: By direct induction on the structure of φ, or by observing that point generation always results in bisimilar models.

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This idea is correct. How do we prove it?

Nominal-free sentences are invariant under generated submodels

Lemma: $\mathcal{M}$ be a model, let $\mathcal{M}_w$ be any of its point generated submodels, and g be an assignment sending all state variables to points in $\mathcal{M}_w$. Then for any nominal-free formula φ (in the hybrid language with $\downarrow$) and any point u in $\mathcal{M}_w$

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Corollary: The truth of pure nominal free sentences is invariant under generated submodels.

What about first-order formulas?

- A first-order formula in one free variable $\varphi(x)$ is invariant under point generated submodels if for any model $\mathcal{M}$, any of its point generated submodels $\mathcal{M}_w$, and any point u in $\mathcal{M}_w$, $\mathcal{M} \models \varphi[u]$ iff $\mathcal{M}' \models \varphi[u]$.
- Obviously not all first-order formulas are invariant under generated submodels — first-order logic is clearly non-local!
- But some are. Which ones? That is, what is the first-order logic of locality?

The logic of locality

Theorem: A first-order formula in one free variable is invariant for generated submodels iff it is equivalent to the standard translation of a nominal-free sentence (of the hybrid language with downarrow).

That is, hybrid logic with downarrow is precisely the first-order logic of locality.

For the original proof see "Hybrid Logics: Characterization, Interpolation and Complexity", Areces, Blackburn and Marx, *Journal of Symbolic Logic*, 66:977-1009, 2001.

For an even better proof see Balder ten Cate's 2004 Amsterdam PhD thesis, *Model Theory for Extended Modal Languages*. (We may discuss this proof on Friday.)

Interpolation

A logic has the interpolation property if whenever

$$\models \varphi \to \psi$$

then there is some formula θ containing only non-logical symbols common to φ and ψ such that:

$$\models \varphi \to \theta \quad \text{and} \quad \models \theta \to \psi.$$

Roughly speaking, if a logic enjoys interpolation, then validity can always be 'filtered through' the common information bearing elements of the language.

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- And neither is basic hybrid logic: we'll now see that interpolation fails in the basic hybrid language.
- However we'll immediately be able to 'repair' this failure with $\downarrow$. And in fact, $\downarrow$ can repair systematically repair interpolation failures.

Interpolation failure in basic hybrid logic

Interpolation failure in basic hybrid logic

In Lecture 1 we gave a tableau proof of

$$(\Diamond \mathrm{p} \wedge \Diamond \neg \mathrm{p}) \rightarrow (\Box(\mathrm{q} \rightarrow i) \rightarrow \Diamond \neg \mathrm{q}).$$

Hence this formula is valid. So if the basic hybrid language enjoys interpolation then there should exist an interpolating θ such that

$$\models (\Diamond \mathrm{p} \wedge \Diamond \neg \mathrm{p}) \rightarrow \theta \quad \text{and} \quad \theta \rightarrow (\Box(\mathrm{q} \rightarrow i) \rightarrow \Diamond \neg \mathrm{q}).$$

Note that θ must be in the empty language (that is, it must be built up solely from $\top$ and $\bot$) as $\{p\} \cap \{i, q\} = \emptyset$.

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- But it seems impossible to express "I have at least two successors" (in the empty language). And there doesn't seem to be any other candidate.
- And a simple bisimulation argument shows that no interpolant exists.

But what if we also had $\downarrow$ at our disposal?

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- The pure, nominal-free, sentence $\downarrow x.\Diamond\downarrow y.@_x\Diamond\neg y$ says that there are at least two distinct accessible states.
- Intuitively, because $\downarrow$ binds variables, we can say a lot (even in the empty language).
- This suggests that although interpolation fails for the basic hybrid language, it might holds for the richer language containing $\downarrow$. And in fact this is just the way things work out...

Hybrid logic with $\downarrow$ has interpolation

Theorem: Suppose we are working with the hybrid language with $\downarrow$. Then the logic of any class of frames definable by a pure, nominal-free, sentence of this language enjoys interpolation.

Proof:
For a model-theoretic proof (using a Chang and Keisler style construction) see "Hybrid Logics: Characterization, Interpolation and Complexity", Areces, Blackburn and Marx, *Journal of Symbolic Logic*, 66:977-1009, 2001. (You will probably see this proof on Friday.)

For a constructive proof (using tableau) see "Constructive interpolants for every bounded fragment definable hybrid logic", Blackburn and Marx, *Journal of Symbolic Logic*, 68(2), 463-480, 2003.

The finite model property

- A language has the finite model property if any satisfiable formula in the language can be satisfied on a finite model.
- The orthodox propositional modal language has the finite model property, and so does the basic hybrid language.
- Viewed negatively, this means that these languages are too weak to define infinite structures.
- Viewed positively, it means that we never need to bother about with infinite structures when working with these languages.

First-order logic lacks the finite model property

Consider the following first-order formulas:

- $\forall x \neg R(x, x)$ (Irreflexivity)
- $\forall x \exists y R(x, y)$ (Unboundedness)
- $\forall x \forall y (R(x, y) \wedge R(y, z) \rightarrow R(x, z))$ (Transitivity)

Any model for these formulas (for example, the natural numbers under their usual ordering) is called a unbounded strict total order. It is not hard to see that any unbounded strict total order is infinite. So first-order logic lack the finite model property.

Hybrid logic with $\downarrow$ also lacks the finite model property

- More difficult to prove, for we lack the globality of first-order logic.
- However we can show this using a spypoint argument.
- We shall define a certain sentence and show that all models satisfying it contain a point s (the spypoint) that can see strict unbounded total order.

A spypoint argument

Consider what any model of the following formula must contain:

$$@_s\Box\Box \downarrow x.@_s\Diamond x$$

$$\wedge \quad @_s\Diamond\neg s$$

$$\wedge \quad @_s\Box\Diamond\top$$

$$\wedge \quad @_s\Box \downarrow x.\neg\Diamond x$$

$$\wedge \quad @_s\Box \downarrow x.\Box\Box \downarrow y.@_x\Diamond y$$

This formula has some obvious models.

Moreover, any model for this formula must contain a point s such that the set of points B that s is related to is an unbounded strict total order — and hence infinite.

Hybrid logic with $\downarrow$ is undecidable

- We have stepped over an important boundary: adding $\downarrow$ has cost us decidability.
- In fact, even the fragment consisting of pure, nominal-free, @-free sentences is undecidable.
- This can also be proved using a spypoint argument. Basic technique is to use the spypoint as a vantage point surveying a coding of an undecidable problem. (See "Hybrid Logics: Characterization, Interpolation and Complexity", Areces, Blackburn and Marx, *Journal of Symbolic Logic*, 66:977-1009, 2001.)
- You'll probably see this in Thursday's lecture ...

Two comments on undecidability

- One interesting decidable fragment is known. Maarten Marx has shown that the fragment in which $\Box$ never occurs under the scope of $\downarrow$ is decidable (and in fact EXPTIME-complete). This fragment can handle some useful description logic definitions.
- Because downarrow binding is local, we always know which substitutions we have to perform. There is no need for Skolem functions or unification. It may be that theorem provers will perform well on "typical" formulas. The **HyLoRes** prover handles downarrow, and it is hoped to optimize it's performance for this binder.

Summing up ...

- We motivated the idea of binding variables to states locally, and introduced $\downarrow$ which lets us dynamically name the here-and-now.
- By doing this we have captured precisely the first-order logic of locality. Completeness and interpolation results hold for all local logics. Although local, the existence of infinite models can be forced, and the system is undecidable.