Hybrid Logic Tango

University of Buenos Aires
February 2008

Carlos Areces and Patrick Blackburn

TALARIS team
INRIA Nancy Grand Est
France
areces@loria.fr
patrick.blackburn@loria.fr

Lecture 3: 13 February 2008

Lecture 3: Completeness

This lecture has two parts:

- In the first part we prove the fundamental completeness result for ordinary modal logic, using the method of canonical models. We 'll start form scratch, introducing all the needed concepts and results. As we'll see, we can prove a general result, but it has limitations. We conclude by stating the Magic lemma for hybrid logic.
- In the second part we prove completeness for basic hybrid logic using the Henkin method, familiar from completeness proofs for first-order logic. This approach enables us to make us of the Magic Lemma to prove general completeness results rather easily.

Where we are going

- Proving (soundness and) completeness results is basically about linking two different conceptions of logic: the semantic and the syntactic — that is, the model-theoretic and the proof-theoretic.
- So to start with, I'll first be precise about some basic semantic concepts (like validity and local semantic consequence.
- I'll then introduce the key syntactic concepts we will use today, notable normal modal logic.
- After that, I'll be more precise about soundness and completeness really mean; in particular, I'll explain the concept of strong completeness.
- And then we'll be ready to prove completeness via canonical models for basic modal logic ...

Validity

In one form or another, the semantic notion of validity is fundamental to the study of logic. When it comes to modal and hybrid logic, the following two concepts of validity are fundamental:

- A formula ϕ is said to be valid if it is true at any point, in any model whatsoever. For example:

  $$\Box(p \to q) \to (\Box p \to \Box q)$$

  is valid.

- A formula ϕ is said to be valid on a class of frames F, if it is true at any point, in any model that is based on a frame $\mathfrak{F}$ in F. For example:

  $$\Diamond\Diamond p \to \Diamond p$$

  is valid on the class of transitive frames. But it is not valid.

Consequence

But logic is not just about validities, it is also about consequence, about conclusions following from premises. This means the following concept is also important:

> *Let let* F *be a class frames. Let* Σ *and* ϕ *be a set of formulas and a single formula (the premises and conclusion respectively). We say that* ϕ *is a local semantic consequence of* Σ over F *(notation:* $\Sigma \Vdash_{\mathsf{F}} \phi$*) if for all models* $\mathfrak{M}$ *built over frames in* F*, and all points* w *in* $\mathfrak{M}$*, if* $\mathfrak{M}, w \Vdash \Sigma$ *then* $\mathfrak{M}, w \Vdash \phi$ *too.*

For example, suppose that we are working with Tran, the class of transitive frames. Then: $\{\Diamond\Diamond p\} \Vdash_{\mathsf{Tran}} \Diamond p$. On the other hand, $\Diamond p$ is *not* a local semantic consequence of $\{\Diamond\Diamond p\}$ over the class of *all* frames.

From Hilbert systems to normal model logics

And having seen the basic semantic concepts, let's turn to syntax:

- A normal modal logic is simply a set of formulas satisfying certain syntactic closure conditions.
- Which conditions? We will work towards the answer by defining a Hilbert-style axiom system called **K**. **K** is the 'minimal' (or 'weakest') system for reasoning about frames; stronger systems are obtained by adding extra axioms.
- As you will see, the definition of a normal modal logic is a more-or-less immediate abstraction from what is involved in Hilbert-style approaches to modal proof theory.
- We will work in the basic modal language (just one $\Diamond$ and $\Box$,and no nominals or @).

K-proofs

Definition

A **K**-*proof* is a finite sequence of formulas, each of which is an axiom, or follows from one or more earlier items in the sequence by applying a rule of proof. The axioms of **K** are all instances of propositional tautologies plus:

(K) $\Box(p \to q) \to (\Box p \to \Box q)$

(Dual) $\Diamond p \leftrightarrow \neg\Box\neg p$.

The rules of proof of **K** are:

- Modus ponens: given ϕ and $\phi \to \psi$, prove ψ.
- Uniform substitution: given ϕ, prove θ, where θ is obtained from ϕ by uniformly replacing proposition letters in ϕ by arbitrary formulas.
- Generalization: given ϕ, prove $\Box\phi$.

A formula ϕ is **K**-provable if it occurs as the last item of some **K**-proof, and if this is the case we write $\vdash_{\mathbf{K}} \phi$.

Soundness

- All the axioms are validities. That is, they are true at all point in all models.
- All the rules of proof preserve validity.
- That is, the axiom system just define is sound. Any formulas we prove using this system must be valid too.
- That is, if $\vdash_{\mathbf{K}} \phi$ then $\Vdash \phi$.
- Thus soundness is a concept that links the worlds of syntax and semantics. It is essentially a 'no garbage' condition.

Completeness

- Moreover, as we will see, the converse is also true: if a basic modal formula is valid, then it is **K**-provable.
- That is, if $\Vdash \phi$ then $\vdash_{\mathbf{K}} \phi$ then
- That is, **K** is complete with respect to the class of all frames.
- Completeness is also a concept that links the worlds of syntax and semantics. But it is a deeper (and more difficult to prove) condition than soundness.
- Taken together, soundness and completeness mean that **K** generates precisely the valid formulas.

Example

The formula $(\Box p \wedge \Box q) \to \Box(p \wedge q)$ is valid on any frame, so it should be **K**-provable. And indeed, it is. To see this, consider the following sequence of formulas:

1.	$\vdash p \to (q \to (p \wedge q))$	*Tautology*
2.	$\vdash \Box(p \to (q \to (p \wedge q)))$	*Generalization: 1*
3.	$\vdash \Box(p \to q) \to (\Box p \to \Box q)$	*K axiom*
4.	$\vdash \Box(p \to (q \to (p \wedge q))) \to (\Box p \to \Box(q \to (p \wedge q)))$	*Uniform Substitution: 3*
5.	$\vdash \Box p \to \Box(q \to (p \wedge q))$	*Modus Ponens: 2, 4*
6.	$\vdash \Box(q \to (p \wedge q)) \to (\Box q \to \Box(p \wedge q))$	*Uniform Substitution: 3*
7.	$\vdash \Box p \to (\Box q \to \Box(p \wedge q))$	*Propositional Logic: 5, 6*
8.	$\vdash (\Box p \wedge \Box q) \to \Box(p \wedge q)$	*Propositional Logic: 7*

Strictly speaking, this sequence is *not* a **K**-proof – it is a subsequence of the proof consisting of the most important items.

Stronger logics

Stronger logics

For many purposes, **K** is too weak. If we are interested in transitive frames, we would like a proof system which reflects this. For example, we know that $\Diamond\Diamond p \to \Diamond p$ is valid on all transitive frames, so we would want a proof system that generates this formula; **K** does not do this, for $\Diamond\Diamond p \to \Diamond p$ is not valid on all frames.

Stronger logics

For many purposes, **K** is too weak. If we are interested in transitive frames, we would like a proof system which reflects this. For example, we know that $\Diamond\Diamond p \to \Diamond p$ is valid on all transitive frames, so we would want a proof system that generates this formula; **K** does not do this, for $\Diamond\Diamond p \to \Diamond p$ is not valid on all frames.

But we can extend **K** to cope with many such restrictions by adding extra axioms. For example, if we enrich **K** by adding $\Diamond\Diamond p \to \Diamond p$ as an axiom, we obtain the Hilbert system called **K4**, which is sound and complete with respect to the class of all transitive frames (that is, it generates *precisely* the formulas valid on transitive frames).

Stronger logics

For many purposes, **K** is too weak. If we are interested in transitive frames, we would like a proof system which reflects this. For example, we know that $\Diamond\Diamond p \to \Diamond p$ is valid on all transitive frames, so we would want a proof system that generates this formula; **K** does not do this, for $\Diamond\Diamond p \to \Diamond p$ is not valid on all frames.

But we can extend **K** to cope with many such restrictions by adding extra axioms. For example, if we enrich **K** by adding $\Diamond\Diamond p \to \Diamond p$ as an axiom, we obtain the Hilbert system called **K4**, which is sound and complete with respect to the class of all transitive frames (that is, it generates *precisely* the formulas valid on transitive frames).

More generally, given any set of modal formulas Γ, we are free to add them as extra axioms to **K**, thus forming the axiom system $\mathbf{K\Gamma}$. In many interesting cases it is possible to characterize such extensions in terms of frame validity.

Normal Modal Logics

A normal modal logic Λ is a set of formulas that contains all tautologies, $\Box(p \to q) \to (\Box p \to \Box q)$, and $\Diamond p \leftrightarrow \neg\Box\neg p$, and that is closed under *modus ponens*, *uniform substitution* and *generalization*. We call the smallest normal modal logic **K**. This definition is a direct abstraction from the ideas underlying modal Hilbert systems. It throws away all talk of proof sequences and concentrates on what is really essential: the presence of axioms and closure under the rules of proof.

Thinking in terms of normal modal logics . . .

- We will rarely mention Hilbert systems again: we prefer to work with the more abstract notion of normal modal logics.
- For a start, it is simpler to work with the set-theoretical notion of membership than with proof sequences.
- Moreover, thinking in terms of normal model logics leads to interesting territory — for it turns out that there are lots of normal modal logics.

Examples of normal modal logics

Examples of normal modal logics

1. The inconsistent logic (that is, the set of all modal formulas) is a normal logic.

Examples of normal modal logics

1. The inconsistent logic (that is, the set of all modal formulas) is a normal logic.
2. If $\{\Lambda_i \mid i \in I\}$ is a collection of normal logics, then $\bigcap_{i \in I} \Lambda_i$ is a normal logic.

Examples of normal modal logics

1. The inconsistent logic (that is, the set of all modal formulas) is a normal logic.
2. If $\{\Lambda_i \mid i \in I\}$ is a collection of normal logics, then $\bigcap_{i\in I} \Lambda_i$ is a normal logic.
3. If F is any class of frames, then Λ_{F} is a normal logic.

This last example is very important. It shows that many modal logics have a semantic characterisation. This leads to the following generalisation of the concepts of soundness and completeness.

Soundness generalised

Let F be a class of frames. A normal modal logic Λ is sound with respect to F if $\Lambda \subseteq \Lambda_{\mathsf{S}}$.

Equivalently: Λ is sound with respect to F if for all formulas ϕ, and all frames $\mathfrak{F} \in \mathsf{F}$, if $\vdash_\Lambda \phi$ then $\mathfrak{F} \Vdash \phi$.

If Λ is sound with respect to F we sometimes say that F is a class of frames for Λ.

Weak Completeness

What about completeness? Well, as before we have the following concept, which we now call weak completeness:

> *A logic* Λ *is* weakly complete *with respect to* F *if for any formula* ϕ*, if* $\mathsf{F} \Vdash \phi$ *then* $\vdash_\Lambda \phi$*.*

But we are also going to introduce a notion of strong completeness to give us a notion of completeness for consequence. Weak completeness will be a special case of strong completeness.

Deducibility (from premises)

First we need the following *syntactic* concept:

If $\Gamma \cup \{\phi\}$ *is a set of formulas then* ϕ *is deducible in* Λ *from* Γ *(or:* ϕ *is* Λ*-deducible from* Γ*) if* $\vdash_\Lambda \phi$*, or there are formulas* $\psi_1, \ldots, \psi_n \in \Gamma$ *such that*

$$\vdash_\Lambda (\psi_1 \wedge \cdots \wedge \psi_n) \rightarrow \phi.$$

If this is the case we write $\Gamma \vdash_\Lambda \phi$*, if not,* $\Gamma \nvdash_\Lambda \phi$*.*

A set of formulas Γ *is* Λ*-consistent if* $\Gamma \nvdash_\Lambda \bot$*, and* Λ*-inconsistent otherwise.*

Remarks

- It is a simple exercise in propositional logic to check that a set of formulas Γ is Λ-inconsistent if and only if there is a formula ϕ such that $\Gamma \vdash_\Lambda \phi \wedge \neg\phi$ if and only if for all formulas ψ, $\Gamma \vdash_\Lambda \psi$.
- Moreover, Γ is Λ-consistent if and only if every finite subset of Γ is. (That is, our notion of deducibility has the *compactness* property.)
- When Λ is clear from context or irrelevant, we drop explicit references to it and talk simply of "deducibility', 'consistency' and 'inconsistency', and use the notation $\vdash \phi$, $\Gamma \vdash \phi$, and so on.

Strong Completeness

Let F be a class of frames. A logic Λ is strongly complete with respect to F if for any set of formulas $\Gamma \cup \{\phi\}$, if $\Gamma \Vdash_{\mathsf{F}} \phi$ then $\Gamma \vdash_{\Lambda} \phi$. That is, if Γ semantically entails ϕ on F, then ϕ is Λ-deducible from Γ.

Completeness is about model existence

A logic Λ is strongly complete with respect to a class of structures F iff every Λ-consistent set of formulas is satisfiable on some $\mathfrak{F} \in \mathsf{F}$. Λ is weakly complete with respect to a class of structures F iff every Λ-consistent formula is satisfiable on some $\mathfrak{F} \in \mathsf{F}$.

Proof:
To prove the right to left implication we argue by contraposition. Suppose Λ is not strongly complete with respect to F. Thus there is a set of formulas $\Gamma \cup \{\phi\}$ such that $\Gamma \Vdash_{\mathsf{S}} \phi$ but $\Gamma \nvdash_{\Lambda} \phi$. Then $\Gamma \cup \{\neg\phi\}$ is Λ-consistent, but not satisfiable on any structure in F.

The left to right direction is left to you. The result for weak completeness follows from the one for strong completeness.

Maximal consistent sets of formulas

Thus, proving completeness is about building models. And what are we going to build our models out of? Very special sets of formulas:

A set of formulas Γ is maximal Λ-consistent if Γ is Λ-consistent, and any set of formulas properly containing Γ is Λ-inconsistent.

If Γ is a maximal Λ-consistent set of formulas, we say it is a Λ-MCS.

Properties of MCSs

If Λ is a logic and Γ is a Λ-MCS then:

1. Γ is closed under modus ponens: if ϕ, $\phi \to \psi \in \Gamma$, then $\psi \in \Gamma$;
2. $\Lambda \subseteq \Gamma$;
3. for all formulas ϕ: $\phi \in \Gamma$ or $\neg\phi \in \Gamma$;
4. for all formulas ϕ, ψ: $\phi \vee \psi \in \Gamma$ iff $\phi \in \Gamma$ or $\psi \in \Gamma$.

Lindenbaum's Lemma

As MCSare to be our building blocks, it is vital we have enough of them. Do we? Lindenbaum's celebrated lemma says yes:

> *If Σ is a Λ-consistent set of formulas then there is a Λ-MCS Σ^+ such that $\Sigma \subseteq \Sigma^+$.*

Proof: Let $\phi_0, \phi_1, \phi_2, \ldots$ be an enumeration of the formulas of our language. We define the set Σ^+ as the union of a chain of Λ-consistent sets as follows:

$$
\begin{aligned}
\Sigma_0 &= \Sigma, \\
\Sigma_{n+1} &= \begin{cases} \Sigma_n \cup \{\phi_n\}, & \text{if this is } \Lambda\text{-consistent} \\ \Sigma_n \cup \{\neg\phi_n\}, & \text{otherwise} \end{cases} \\
\Sigma^+ &= \bigcup_{n \geq 0} \Sigma_n.
\end{aligned}
$$

The proof of the following properties of Σ^+ is left for you: (i) Σ_n is Λ-consistent, for all n; (ii) exactly one of ϕ and $\neg\phi$ is in Σ^+, for every formula ϕ; (iii) if $\Sigma^+ \vdash_\Lambda \phi$, then $\phi \in \Sigma^+$; and finally (iv) Σ^+ is a Λ-MCS.

Canonical Models

The canonical model $\mathfrak{M}^\Lambda$ for a normal modal logic Λ (in the basic language) is the triple $(W^\Lambda, R^\Lambda, V^\Lambda)$ where:

1. W^Λ is the set of all Λ-MCSs;
2. R^Λ is the binary relation on W^Λ defined by $R^\Lambda wu$ if for all formulas ψ, $\psi \in u$ implies $\Diamond\psi \in w$. R^Λ is called the *canonical relation*;
3. V^Λ is the valuation defined by $V^\Lambda(p) = \{w \in W^\Lambda \mid p \in w\}$. V^Λ is called the *canonical* (or *natural*) *valuation*.

The pair $\mathfrak{F}^\Lambda = (W^\Lambda, R^\Lambda)$ is called the *canonical frame* for Λ.

Note: each normal logic Λ has a different canonical model.

Observation

For any normal logic Λ, $R^{\Lambda}wv$ iff for all formulas ψ, $\Box\psi \in w$ implies $\psi \in v$.

Proof: For the left to right direction, suppose $R^{\Lambda}wv$. Further suppose $\psi \notin v$. As v is an MCS, by the properties of MCSnoted on a previous slide, $\neg\psi \in v$. As $R^{\Lambda}wv$, $\Diamond\neg\psi \in w$. As w is consistent, $\neg\Diamond\neg\psi \notin w$. That is, $\Box\psi \notin w$ and we have established the contrapositive.

We leave the right to left direction to you!

Existence Lemma

For any normal modal logic Λ and any state $w \in W^{\Lambda}$, if $\Diamond\phi \in w$ then there is a state $v \in W^{\Lambda}$ such that $R^{\Lambda}wv$ and $\phi \in v$.

An important lemma, but pretty easy to prove. You can find a proof in pretty much any text on modal logic. The key trick is to show that: $\{\phi\} \cup \{\psi \mid \Box\psi \in w\}$ is consistent. Why?

Truth Lemma

For any normal modal logic Λ and any formula ϕ, $\mathfrak{M}^\Lambda, w \Vdash \phi$ iff $\phi \in w$.

Proof: By induction on the degree of ϕ. The base case follows from the definition of V^Λ. The boolean cases follow from the properties of MCSnoted earlier. It remains to deal with the modalities. The left to right direction is more or less immediate from the definition of R^Λ:

$$
\begin{array}{lll l}
\mathfrak{M}^\Lambda, w \Vdash \Diamond\phi & \text{iff} & \exists v\,(R^\Lambda wv \;\wedge\; \mathfrak{M}^\Lambda, v \Vdash \phi) & \\
 & \text{iff} & \exists v\,(R^\Lambda wv \;\wedge\; \phi \in v) & \text{(Induct Hyp)} \\
 & \text{only if} & \Diamond\phi \in w & \text{(Definition } R^\Lambda\text{).}
\end{array}
$$

For the right to left direction, suppose $\Diamond\phi \in w$. By the equivalences above, it suffices to find an MCS v such that $R^\Lambda wv$ and $\phi \in v$ – and this is precisely what the Existence Lemma guarantees.

Canonical Model theorem

Any normal modal logic is strongly complete with respect to its canonical model.

Proof: Suppose Σ is a consistent set of the normal modal logic Λ. By Lindenbaum's Lemma there is a Λ-MCS Σ^+ extending Σ. By the previous lemma, $\mathfrak{M}^\Lambda, \Sigma^+ \Vdash \Sigma$.

Now, this is certainly general, and it is certainly nice, but it is abstract. In particular, what does it tell us about completeness with respect to a class of frames?

Corollaries

- **K** is strongly complete with respect to the class of all frames.
- **K4** is strongly complete with respect to the class of transitive frames.

But getting really general results is hard. The best known general result in this area is the Sahlqvist Completeness Theorem, which says that completeness is automatic with respect to the intended class of frames if our axioms are all Sahlqvist formulas. This is nice, but not easy to prove or apply,...

So what about hybrid logic?

- In fact, we're not going to use the canonical Model approach.
- We will use the idea of building models from MCS, and we will have a Lindenbaum's Lemma, and much will be the same.
- But our approach will be much more first-order in spirit. In fact. in essence we will do a Henkin Completeness Proof.
- And our basic completeness result will be much stronger than the Canonical Model Theorem. We're going to get lots of frame completeness results right away. Why? Because of the Magic Lemma ...

Magic Lemma

If ϕ is a pure formula, we say that ψ is a pure instance of ϕ if ψ is obtained from ϕ by uniformly substituting nominals for nominals. Then we have:

Let $\mathfrak{M} = (\mathfrak{F}, V)$ be a named model and ϕ a pure formula. Suppose that for all pure instances ψ of ϕ, and all $w \in \mathfrak{F}$, $\mathfrak{M}, w \Vdash \psi$. Then $\mathfrak{F} \Vdash \phi$.

Normal Hybrid Logics

Definition

A set of formulas Λ in the basic hybrid language is a normal hybrid logic if it contains:

- All tautologies,
- $\Box(p \to q) \to (\Box p \to \Box q)$
- $\Diamond p \leftrightarrow \neg\Box\neg p$
- and all the axioms listed on the following slides

is closed under the following rules of proof:

- modus ponens and generalization,
- $@_i$-generalization (if ϕ is provable, so is $@_i\phi$, for any i)
- sorted substitution (if $\phi \in \Lambda$, and θ results from ϕ by uniformly replacing proposition letters by arbitrary formulas, and nominals by nominals, then $\theta \in \Lambda$).

We call the smallest normal hybrid logic $\mathbf{K}_h$.

Axioms controlling logic of @

($K_{@}$)	$@_i(p \to q) \to (@_i p \to @_i q)$,
(*self-dual*)	$@_i p \leftrightarrow \neg @_i \neg p$,
(*introduction*)	$i \wedge p \to @_i p$.

As satisfaction operators are normal modal operators, the inclusion of $K_{@}$ should come as no surprise. As for self-dual, note that self-dual modalities are those whose transition relation is a *function*: given the jump-to-the-named-state interpretation of satisfaction operators, this is exactly the axiom we would expect. Introduction tells us how to place information under the scope of satisfaction operators. Actually, it also tells us how to get hold of such information, for if we replace p by $\neg p$, contrapose, and make use of *self-dual*, we obtain $(i \wedge @_i p) \to p$; we call this the elimination formula.

Axioms controlling state equality

$$\begin{array}{ll} (\mathit{ref}) & @_i i, \\ (\mathit{sym}) & @_i j \leftrightarrow @_j i, \\ (\mathit{nom}) & @_i j \wedge @_j p \rightarrow @_i p, \\ (\mathit{agree}) & @_j @_i p \leftrightarrow @_i p. \end{array}$$

Note that the transitivity of naming follows from the *nom* axiom; for example, substituting the nominal k for the propositional variable p yields $@_i j \wedge @_j k \rightarrow @_i k$.

Axiom controlling the interaction between $\Diamond$ and @

($back$) $\Diamond @_i p \rightarrow @_i p$.

Note that $\Diamond i \wedge @_i p \rightarrow \Diamond p$ is another valid @-$\Diamond$ interaction principle; it is called bridge and we will use it when we prove the Existence Lemma. However *bridge* is provable in $\mathbf{K}_h$ — left as an Exercise!

Some useful concepts

Let us say that a $\mathbf{K}_h$-MCS is named if and only if it contains a nominal, and call any nominal belonging to a $\mathbf{K}_h$-MCS a name for that MCS.

Now, $\mathbf{K}_h$ is strong enough to prove a lemma which is fundamental to our later work: hidden inside any $\mathbf{K}_h$-MCS are a collection of named MCSs with a number of desirable properties…

Nice properties of named MCSs

Lemma (Yield Lemma)

Let Γ *be a* $\mathbf{K}_h$-MCS*. For every nominal* i*, let* Δ_i *be* $\{\phi \mid @_i\phi \in \Gamma\}$*. Then:*

1. *For every nominal* i*,* Δ_i *is a* $\mathbf{K}_h$-MCS *that contains* i*.*
2. *For all nominals* i *and* j*, if* $i \in \Delta_j$*, then* $\Delta_j = \Delta_i$*.*
3. *For all nominals* i *and* j*,* $@_i\phi \in \Delta_j$ *iff* $@_i\phi \in \Gamma$*.*
4. *If* k *is a name for* Γ*, then* $\Gamma = \Delta_k$*.*

Proof of property (i)

For every nominal i, Δ_i is a $\mathbf{K}_h$-MCS that contains i.

First, for every nominal i we have the *ref* axiom $@_i i$, hence $i \in \Delta_i$. Next, Δ_i is consistent. For assume for the sake of a contradiction that it is not. Then there are $\delta_1, ..., \delta_n \in \Delta_i$ such that $\vdash \neg(\delta_1 \wedge \cdots \wedge \delta_n)$. By $@_i$-necessitation, $\vdash @_i \neg(\delta_1 \wedge \cdots \wedge \delta_n)$, hence $@_i \neg(\delta_1 \wedge \cdots \wedge \delta_n)$ is in Γ, and thus by *self-dual* $\neg @_i(\delta_1 \wedge \cdots \wedge \delta_n)$ is in Γ too. On the other hand, as $\delta_1, ..., \delta_n \in \Delta_i$, we have $@_i\delta_1, ..., @_i\delta_n \in \Gamma$. As $@_i$ is a normal modality, $@_i(\delta_1 \wedge \cdots \wedge \delta_n) \in \Gamma$ as well, contradicting the consistency of Γ. So Δ_i is consistent.

Is Δ_i maximal? Assume it is not. Then there is a formula χ such that neither χ nor $\neg\chi$ is in Δ_i. But then both $\neg @_i \chi$ and $\neg @_i \neg \chi$ belong to Γ, and this is impossible: if $\neg @_i \chi \in \Gamma$, then by self-duality $@_i \neg \chi \in \Gamma$ as well. We conclude that Δ_i is a $\mathbf{K}_h$-MCS named by i.

Proof of property (ii)

For all nominals i and j, if $i \in \Delta_j$, then $\Delta_j = \Delta_i$.

Suppose $i \in \Delta_j$; we will show that $\Delta_j = \Delta_i$. As $i \in \Delta_j$, $@_j i \in \Gamma$. Hence, by *sym*, $@_i j \in \Gamma$ too. But now the result is more-or-less immediate. First, $\Delta_j \subseteq \Delta_i$. For if $\phi \in \Delta_j$, then $@_j \phi \in \Gamma$. Hence, as $@_i j \in \Gamma$, it follows by *nom* that $@_i \phi \in \Gamma$, and hence that $\phi \in \Delta_i$ as required. A similar *nom*-based argument shows that $\Delta_i \subseteq \Delta_j$ — check it out!

Proof of property (iii)

For all nominals i and j, $@_i\phi \in \Delta_j$ iff $@_i\phi \in \Gamma$.

By definition $@_i\phi \in \Delta_j$ iff $@_j@_i\phi \in \Gamma$. By *agree*, $@_j@_i\phi \in \Gamma$ iff $@_i\phi \in \Gamma$. (We call this the *@-agreement property*; it plays an important role in the completeness proof.)

Proof of property (iv)

If k is a name for Γ, then $\Gamma = \Delta_k$.

Suppose Γ is named by k. Let $\phi \in \Gamma$. Then as $k \in \Gamma$, by *introduction* $@_k\phi \in \Gamma$, and hence $\phi \in \Delta_k$. Conversely, if $\phi \in \Delta_k$, then $@_k\phi \in \Gamma$. Hence, as $k \in \Gamma$, by *elimination* we have $\phi \in \Gamma$.

Definition

In what follows, if Γ is a $\mathbf{K}_h$-MCS and i is a nominal, then we will call $\{\phi \mid @_i\phi \in \Gamma\}$ a named set yielded by Γ.

The lemma we have just proved tells us that yielded sets have lots of nice properties.

And now we need to start thinking ...

The crossroads

We have reached an important crossroad. It is now reasonably straightforward to prove that $\mathbf{K}_h$ is the minimal hybrid logic. We would do so as follows. Given a $\mathbf{K}_h$-consistent set of sentences Σ, use the ordinary Lindenbaum's Lemma to expand it to a $\mathbf{K}_h$-MCS Σ^+, and build a model by taking the submodel of the ordinary canonical model generated by

$$\Sigma^+ \cup \{\Delta_i \mid \Delta_i \text{ is a named set yielded by } \Sigma^+\}.$$

Try it out!

Our ambition

But we have a more ambitious goal in mind: we do not want to build just any model, we want a named model. After all, this will enable us to apply the Magic Lemma to prove the completeness for any pure axiomatic extension!

Nice idea! But we face two problems. The first is this. Given a $\mathbf{K}_h$-consistent set of formula, we can certainly expand it to an MCS using Lindenbaum's Lemma – but nothing we have seen so far guarantees that this MCS will be named.

The second problem

The second problem is much deeper. Suppose we overcame the first problem and learned how to expand any consistent set of sentences Σ to a named MCS Σ^+. Now, as we want to build a named model, this pretty much dictates that only the named MCSs yielded by Σ^+ should be used in the model construction. And now for the tough part: nothing we have seen so far guarantees that there are enough MCSs here to support an Existence Lemma.

Incidentally, note that the completeness-via-generation method that I encouraged you to prove a couple of slides back does *not* face this problem: generation automatically gives us all successor MCSs, so we can make use of the ordinary modal Existence Lemma. Unfortunately, not all these successor MCSs need be named, so the generation method will not help with the stronger result wee have in mind. That is, the completeness-via-generation method only gives completeness for the minimal logic — not for all pure extensions.

The solution: add new rules

Here are the stars of the show ...

$$(\textsc{name})\ \frac{\vdash j \rightarrow \theta}{\vdash \theta} \qquad\qquad (\textsc{paste})\ \frac{\vdash @_i \Diamond j \wedge @_j \phi \rightarrow \theta}{\vdash @_i \Diamond \phi \rightarrow \theta}$$

In both rules, j is a nominal distinct from i that does not occur in ϕ or θ.

The NAME rule is going to solve our first problem, the PASTE rule our second.

There is a *lot* more that is worth saying about these two rules, and we'll say some of it later, but for now let's concentrate on putting them to work ...

Some preliminary definitions

Let $\mathbf{K}_h + \textsc{rules}$ be the logic obtained by adding the NAME and PASTE rules to $\mathbf{K}_h$. We say that a $\mathbf{K}_h + \textsc{rules-mcs}$ Γ is pasted iff $@_i\Diamond\phi \in \Gamma$ implies that for some nominal j, $@_i\Diamond j \wedge @_j\phi \in \Gamma$.

And now for the key observation: our new rules guarantee we can extend any $\mathbf{K}_h + \textsc{rules}$-consistent set of sentences to a *named and pasted* $\mathbf{K}_h + \textsc{Rules-mcs}$, provided we enrich the language with new nominals.

We are moving towards a another way of making models, the Henkin method, familiar from first-order logic.

Recall that Henkin completeness proofs for first-order we always to add new constants to the language — and nominals translate into constants

Extended Lindenbaum Lemma

Lemma (Extended Lindenbaum Lemma)

Let Ω' be a (countably) infinite collection of nominals disjoint from Ω, and let $\mathcal{L}'$ be the language obtained by adding these new nominals to $\mathcal{L}$. Then every $\mathbf{K}_h +$ RULES-consistent set of formulas in language $\mathcal{L}$ can be extended to a named and pasted $\mathbf{K}_h +$ RULES-MCS in language $\mathcal{L}'$.

Extended Lindenbaum Lemma

Lemma (Extended Lindenbaum Lemma)

Let Ω' be a (countably) infinite collection of nominals disjoint from Ω, and let $\mathcal{L}'$ be the language obtained by adding these new nominals to $\mathcal{L}$. Then every $\mathbf{K}_h + \textsc{rules}$-consistent set of formulas in language $\mathcal{L}$ can be extended to a named and pasted $\mathbf{K}_h + \textsc{Rules-mcs}$ in language $\mathcal{L}'$.

Proof: Enumerate Ω'. Given a consistent set of $\mathcal{L}$-formulas Σ, define Σ_k to be $\Sigma \cup \{k\}$, where k is the first new nominal in our enumeration.

Extended Lindenbaum Lemma

Lemma (Extended Lindenbaum Lemma)

Let Ω' *be a (countably) infinite collection of nominals disjoint from* Ω*, and let* $\mathcal{L}'$ *be the language obtained by adding these new nominals to* $\mathcal{L}$*. Then every* $\mathbf{K}_h + \textsc{rules}$*-consistent set of formulas in language* $\mathcal{L}$ *can be extended to a named and pasted* $\mathbf{K}_h + \textsc{Rules-mcs}$ *in language* $\mathcal{L}'$*.*

Proof: Enumerate Ω'. Given a consistent set of $\mathcal{L}$-formulas Σ, define Σ_k to be $\Sigma \cup \{k\}$, where k is the first new nominal in our enumeration.

Σ_k is consistent.

Extended Lindenbaum Lemma

Lemma (Extended Lindenbaum Lemma)

Let Ω' be a (countably) infinite collection of nominals disjoint from Ω, and let $\mathcal{L}'$ be the language obtained by adding these new nominals to $\mathcal{L}$. Then every $\mathbf{K}_h + \textsc{rules}$-consistent set of formulas in language $\mathcal{L}$ can be extended to a named and pasted $\mathbf{K}_h + \textsc{Rules-mcs}$ in language $\mathcal{L}'$.

Proof: Enumerate Ω'. Given a consistent set of $\mathcal{L}$-formulas Σ, define Σ_k to be $\Sigma \cup \{k\}$, where k is the first new nominal in our enumeration.

Σ_k is consistent.For suppose not. Then for some conjunction of formulas θ from Σ, $\vdash k \to \neg\theta$. But as k is a new nominal, it does not occur in θ; hence, by the NAME rule, $\vdash \neg\theta$. But this contradicts the consistency of Σ, so Σ_k is consistent after all.

Extended Lindenbaum Lemma

Lemma (Extended Lindenbaum Lemma)

Let Ω' be a (countably) infinite collection of nominals disjoint from Ω, and let $\mathcal{L}'$ be the language obtained by adding these new nominals to $\mathcal{L}$. Then every $\mathbf{K}_h + \textsc{rules}$-consistent set of formulas in language $\mathcal{L}$ can be extended to a named and pasted $\mathbf{K}_h + \textsc{Rules-mcs}$ in language $\mathcal{L}'$.

Proof: Enumerate Ω'. Given a consistent set of $\mathcal{L}$-formulas Σ, define Σ_k to be $\Sigma \cup \{k\}$, where k is the first new nominal in our enumeration.

Σ_k is consistent. For suppose not. Then for some conjunction of formulas θ from Σ, $\vdash k \rightarrow \neg\theta$. But as k is a new nominal, it does not occur in θ; hence, by the NAME rule, $\vdash \neg\theta$. But this contradicts the consistency of Σ, so Σ_k is consistent after all.

So far so good. We've named our original MCS…

Proof of Extended Lindenbaum Lemma continued

Proof of Extended Lindenbaum Lemma continued

We now paste.

Proof of Extended Lindenbaum Lemma continued

We now paste. Enumerate all the formulas of $\mathcal{L}'$, define Σ^0 to be Σ_k, and suppose we have defined Σ^m, where $m \geq 0$. Let ϕ_{m+1} be the $(m+1)$-th formula in our enumeration of $\mathcal{L}'$. We define Σ^{m+1} as follows:

Proof of Extended Lindenbaum Lemma continued

We now paste. Enumerate all the formulas of $\mathcal{L}'$, define Σ^0 to be Σ_k, and suppose we have defined Σ^m, where $m \geq 0$. Let ϕ_{m+1} be the $(m+1)$-th formula in our enumeration of $\mathcal{L}'$. We define Σ^{m+1} as follows: If $\Sigma^{m+1} \cup \{\phi_{m+1}\}$ is inconsistent, then $\Sigma^{m+1} = \Sigma^m$.

Proof of Extended Lindenbaum Lemma continued

We now paste. Enumerate all the formulas of $\mathcal{L}'$, define Σ^0 to be Σ_k, and suppose we have defined Σ^m, where $m \geq 0$. Let ϕ_{m+1} be the $(m+1)$-th formula in our enumeration of $\mathcal{L}'$. We define Σ^{m+1} as follows: If $\Sigma^{m+1} \cup \{\phi_{m+1}\}$ is inconsistent, then $\Sigma^{m+1} = \Sigma^m$. Otherwise:

Proof of Extended Lindenbaum Lemma continued

We now paste. Enumerate all the formulas of $\mathcal{L}'$, define Σ^0 to be Σ_k, and suppose we have defined Σ^m, where $m \geq 0$. Let ϕ_{m+1} be the $(m+1)$-th formula in our enumeration of $\mathcal{L}'$. We define Σ^{m+1} as follows: If $\Sigma^{m+1} \cup \{\phi_{m+1}\}$ is inconsistent, then $\Sigma^{m+1} = \Sigma^m$. Otherwise:

1. $\Sigma^{m+1} = \Sigma^m \cup \{\phi_{m+1}\}$ if ϕ_{m+1} is not of the form $@_i\Diamond\phi$. (Here i can be any nominal.)

Proof of Extended Lindenbaum Lemma continued

We now paste. Enumerate all the formulas of $\mathcal{L}'$, define Σ^0 to be Σ_k, and suppose we have defined Σ^m, where $m \geq 0$. Let ϕ_{m+1} be the $(m+1)$-th formula in our enumeration of $\mathcal{L}'$. We define Σ^{m+1} as follows: If $\Sigma^{m+1} \cup \{\phi_{m+1}\}$ is inconsistent, then $\Sigma^{m+1} = \Sigma^m$. Otherwise:

1. $\Sigma^{m+1} = \Sigma^m \cup \{\phi_{m+1}\}$ if ϕ_{m+1} is not of the form $@_i\Diamond\phi$. (Here i can be any nominal.)
2. $\Sigma^{m+1} = \Sigma^m \cup \{\phi_{m+1}\} \cup \{@_i\Diamond j \wedge @_j\phi\}$, if ϕ_{m+1} is of the form $@_i\Diamond\phi$. (Here j is the first nominal in the new nominal enumeration that does not occur in Σ^m or $@_i\Diamond\phi$.)

Proof of Extended Lindenbaum Lemma continued

We now paste. Enumerate all the formulas of $\mathcal{L}'$, define Σ^0 to be Σ_k, and suppose we have defined Σ^m, where $m \geq 0$. Let ϕ_{m+1} be the $(m+1)$-th formula in our enumeration of $\mathcal{L}'$. We define Σ^{m+1} as follows: If $\Sigma^{m+1} \cup \{\phi_{m+1}\}$ is inconsistent, then $\Sigma^{m+1} = \Sigma^m$. Otherwise:

1. $\Sigma^{m+1} = \Sigma^m \cup \{\phi_{m+1}\}$ if ϕ_{m+1} is not of the form $@_i\Diamond\phi$. (Here i can be any nominal.)
2. $\Sigma^{m+1} = \Sigma^m \cup \{\phi_{m+1}\} \cup \{@_i\Diamond j \wedge @_j\phi\}$, if ϕ_{m+1} is of the form $@_i\Diamond\phi$. (Here j is the first nominal in the new nominal enumeration that does not occur in Σ^m or $@_i\Diamond\phi$.)

Henkin territory! In a first-order Henkin proofs we use constants to witness existential statements; here we use nominals to witness diamond statements.

Proof of Extended Lindenbaum Lemma continued

Proof of Extended Lindenbaum Lemma continued

Let $\Sigma^+ = \bigcup_{n \geq 0} \Sigma^n$. Clearly this set is named, maximal, and pasted.

Proof of Extended Lindenbaum Lemma continued

Let $\Sigma^+ = \bigcup_{n \geq 0} \Sigma^n$. Clearly this set is named, maximal, and pasted.

Furthermore, it is consistent, for the only non-trivial aspects of the expansion is that defined by the second item, and the consistency of this step is precisely what the PASTE rule guarantees.

Proof of Extended Lindenbaum Lemma continued

Let $\Sigma^+ = \bigcup_{n \geq 0} \Sigma^n$. Clearly this set is named, maximal, and pasted.

Furthermore, it is consistent, for the only non-trivial aspects of the expansion is that defined by the second item, and the consistency of this step is precisely what the PASTE rule guarantees.

So we have proved the Extended Lindenbaum Lemma we require

Proof of Extended Lindenbaum Lemma continued

Let $\Sigma^+ = \bigcup_{n \geq 0} \Sigma^n$. Clearly this set is named, maximal, and pasted.

Furthermore, it is consistent, for the only non-trivial aspects of the expansion is that defined by the second item, and the consistency of this step is precisely what the PASTE rule guarantees.

So we have proved the Extended Lindenbaum Lemma we require

Note the similarity of this argument to the standard completeness proof for first-order logic: in essence, PASTE gives us the deductive power required to use nominals as Henkin constants.

Defining models

And now we can define the models we need. We are basically going to use the named sets examined in Lemma 3 (the Yield Lemma) but with one small but crucial change: instead of starting with an arbitrary $\mathbf{K}_h$-MCS, we will insist on using the named sets yielded by a named and pasted $\mathbf{K}_h + \textsc{Rules}$-MCS.

The Henkin Model

Let Γ be a named and pasted $\mathbf{K}_h + \textsc{Rules}$-MCS. The named model yielded by Γ, is $\mathfrak{M}^\Gamma = (W^\Gamma, R^\Gamma, V^\Gamma)$.

Here W^Γ is the set of all named sets yielded by Γ, R is the restriction to W^Γ of the usual canonical relation between MCSs (so $R^\Gamma uv$ iff for all formulas ϕ, $\phi \in v$ implies $\Diamond\phi \in u$) and V^Γ is the usual canonical valuation (so for any atom a, $V^\Gamma(a) = \{w \in W^\Gamma \mid a \in w\}$).

Remark: Note that $\mathfrak{M}^\Gamma$ really is a *model*: by items (i) and (ii) of the Yield Lemma, V^Γ assigns every nominal a *singleton* subset of W^Γ.

And now, because we insisted that Γ be named and pasted, we *can* prove the Existence Lemma we require . . .

Existence Lemma

Lemma (Existence Lemma)

Let Γ *be a named and pasted* $\mathbf{K}_h + \textsc{rules-mcs}$*, and let* $\mathfrak{M} = (W, R, V)$ *be the named model yielded by* Γ*. Suppose* $u \in W$ *and* $\Diamond\phi \in u$*. Then there is a* $v \in W$ *such that* Ruv *and* $\phi \in v$*.*

Existence Lemma

Lemma (Existence Lemma)

Let Γ *be a named and pasted* $\mathbf{K}_h + \textsc{rules-mcs}$*, and let* $\mathfrak{M} = (W, R, V)$ *be the named model yielded by* Γ*. Suppose* $u \in W$ *and* $\Diamond\phi \in u$*. Then there is a* $v \in W$ *such that* Ruv *and* $\phi \in v$*.*

Proof: As $u \in W$, for some nominal i we have that $u = \Delta_i$.

Existence Lemma

Lemma (Existence Lemma)

Let Γ *be a named and pasted* $\mathbf{K}_h + \textsc{rules-mcs}$*, and let* $\mathfrak{M} = (W, R, V)$ *be the named model yielded by* Γ*. Suppose* $u \in W$ *and* $\Diamond\phi \in u$*. Then there is a* $v \in W$ *such that* Ruv *and* $\phi \in v$*.*

Proof: As $u \in W$, for some nominal i we have that $u = \Delta_i$. Hence as $\Diamond\phi \in u$, $@_i\Diamond\phi \in \Gamma$.

Existence Lemma

Lemma (Existence Lemma)

Let Γ be a named and pasted $\mathbf{K}_h + \text{RULES-MCS}$, and let $\mathfrak{M} = (W, R, V)$ be the named model yielded by Γ. Suppose $u \in W$ and $\Diamond\phi \in u$. Then there is a $v \in W$ such that Ruv and $\phi \in v$.

Proof: As $u \in W$, for some nominal i we have that $u = \Delta_i$. Hence as $\Diamond\phi \in u$, $@_i\Diamond\phi \in \Gamma$. But Γ is pasted so for some nominal j, $@_i\Diamond j \wedge @_j\phi \in \Gamma$.

Existence Lemma

Lemma (Existence Lemma)

Let Γ *be a named and pasted* $\mathbf{K}_h + \textsc{rules-mcs}$*, and let* $\mathfrak{M} = (W, R, V)$ *be the named model yielded by* Γ*. Suppose* $u \in W$ *and* $\Diamond\phi \in u$*. Then there is a* $v \in W$ *such that* Ruv *and* $\phi \in v$*.*

Proof: As $u \in W$, for some nominal i we have that $u = \Delta_i$. Hence as $\Diamond\phi \in u$, $@_i\Diamond\phi \in \Gamma$. But Γ is pasted so for some nominal j, $@_i\Diamond j \wedge @_j\phi \in \Gamma$. And so $\Diamond j \in \Delta_i$ and $\phi \in \Delta_j$.

Existence Lemma

Lemma (Existence Lemma)

Let Γ be a named and pasted $\mathbf{K}_h + \text{RULES-MCS}$, and let $\mathfrak{M} = (W, R, V)$ be the named model yielded by Γ. Suppose $u \in W$ and $\Diamond\phi \in u$. Then there is a $v \in W$ such that Ruv and $\phi \in v$.

Proof: As $u \in W$, for some nominal i we have that $u = \Delta_i$. Hence as $\Diamond\phi \in u$, $@_i\Diamond\phi \in \Gamma$. But Γ is pasted so for some nominal j, $@_i\Diamond j \wedge @_j\phi \in \Gamma$. And so $\Diamond j \in \Delta_i$ and $\phi \in \Delta_j$.

Hence, if we could show that $R\Delta_i\Delta_j$, then Δ_j would be a suitable choice of v.

Existence Lemma

Lemma (Existence Lemma)

Let Γ *be a named and pasted* $\mathbf{K}_h + \textsc{rules-mcs}$*, and let* $\mathfrak{M} = (W, R, V)$ *be the named model yielded by* Γ*. Suppose* $u \in W$ *and* $\Diamond\phi \in u$*. Then there is a* $v \in W$ *such that* Ruv *and* $\phi \in v$*.*

Proof: As $u \in W$, for some nominal i we have that $u = \Delta_i$. Hence as $\Diamond\phi \in u$, $@_i\Diamond\phi \in \Gamma$. But Γ is pasted so for some nominal j, $@_i\Diamond j \wedge @_j\phi \in \Gamma$. And so $\Diamond j \in \Delta_i$ and $\phi \in \Delta_j$.

Hence, if we could show that $R\Delta_i\Delta_j$, then Δ_j would be a suitable choice of v. So suppose $\psi \in \Delta_j$.

Existence Lemma

Lemma (Existence Lemma)

Let Γ be a named and pasted $\mathbf{K}_h + \textsc{rules}$-$\textsc{mcs}$, and let $\mathfrak{M} = (W, R, V)$ be the named model yielded by Γ. Suppose $u \in W$ and $\Diamond\phi \in u$. Then there is a $v \in W$ such that Ruv and $\phi \in v$.

Proof: As $u \in W$, for some nominal i we have that $u = \Delta_i$. Hence as $\Diamond\phi \in u$, $@_i\Diamond\phi \in \Gamma$. But Γ is pasted so for some nominal j, $@_i\Diamond j \wedge @_j\phi \in \Gamma$. And so $\Diamond j \in \Delta_i$ and $\phi \in \Delta_j$.

Hence, if we could show that $R\Delta_i\Delta_j$, then Δ_j would be a suitable choice of v. So suppose $\psi \in \Delta_j$. This means that $@_j\psi \in \Gamma$. By @-agreement (item (iii) of the Yield Lemma $@_j\psi \in \Delta_i$.

Existence Lemma

Lemma (Existence Lemma)

Let Γ *be a named and pasted* $\mathbf{K}_h + \textsc{rules-mcs}$*, and let* $\mathfrak{M} = (W, R, V)$ *be the named model yielded by* Γ*. Suppose* $u \in W$ *and* $\Diamond\phi \in u$*. Then there is a* $v \in W$ *such that* Ruv *and* $\phi \in v$*.*

Proof: As $u \in W$, for some nominal i we have that $u = \Delta_i$. Hence as $\Diamond\phi \in u$, $@_i\Diamond\phi \in \Gamma$. But Γ is pasted so for some nominal j, $@_i\Diamond j \wedge @_j\phi \in \Gamma$. And so $\Diamond j \in \Delta_i$ and $\phi \in \Delta_j$.

Hence, if we could show that $R\Delta_i\Delta_j$, then Δ_j would be a suitable choice of v. So suppose $\psi \in \Delta_j$. This means that $@_j\psi \in \Gamma$. By @-agreement (item (iii) of the Yield Lemma $@_j\psi \in \Delta_i$. But $\Diamond j \in \Delta_i$. Hence, by *bridge*, $\Diamond\psi \in \Delta_i$ as required.

Truth Lemma

Lemma (Truth Lemma)

Let $\mathfrak{M} = (W, R, V)$ *be the named model yielded by a named and pasted* $\mathbf{K}_h + \textsc{rules-mcs}\ \Gamma$*, and let* $u \in W$*. Then, for all formulas* ϕ*,* $\phi \in u$ *iff* $\mathfrak{M}, u \Vdash \phi$*.*

Truth Lemma

Lemma (Truth Lemma)

Let $\mathfrak{M} = (W, R, V)$ *be the named model yielded by a named and pasted* $\mathbf{K}_h + \textsc{rules-mcs}\ \Gamma$*, and let* $u \in W$*. Then, for all formulas* ϕ*,* $\phi \in u$ *iff* $\mathfrak{M}, u \Vdash \phi$*.*

Proof: Induction on the structure of ϕ.

Truth Lemma

Lemma (Truth Lemma)

Let $\mathfrak{M} = (W, R, V)$ be the named model yielded by a named and pasted $\mathbf{K}_h + \textsc{rules-mcs}\ \Gamma$, and let $u \in W$. Then, for all formulas ϕ, $\phi \in u$ iff $\mathfrak{M}, u \Vdash \phi$.

Proof: Induction on the structure of ϕ. The atomic, boolean, and modal cases are obvious (we use the Existence Lemma just proved for the modalities).

Truth Lemma

Lemma (Truth Lemma)

Let $\mathfrak{M} = (W, R, V)$ *be the named model yielded by a named and pasted* $\mathbf{K}_h + \textsc{rules-mcs}\ \Gamma$*, and let* $u \in W$*. Then, for all formulas* ϕ*,* $\phi \in u$ *iff* $\mathfrak{M}, u \Vdash \phi$*.*

Proof: Induction on the structure of ϕ. The atomic, boolean, and modal cases are obvious (we use the Existence Lemma just proved for the modalities). What about the satisfaction operators?

Truth Lemma

Lemma (Truth Lemma)

Let $\mathfrak{M} = (W, R, V)$ be the named model yielded by a named and pasted $\mathbf{K}_h$ + RULES-MCS Γ, and let $u \in W$. Then, for all formulas ϕ, $\phi \in u$ iff $\mathfrak{M}, u \Vdash \phi$.

Proof: Induction on the structure of ϕ. The atomic, boolean, and modal cases are obvious (we use the Existence Lemma just proved for the modalities). What about the satisfaction operators? Suppose $\mathfrak{M}, u \Vdash @_i\psi$. This happens iff $\mathfrak{M}, \Delta_i \Vdash \psi$ (for by items (i) and (ii) of the Yield Lemma, Δ_i is the only MCS containing i, and hence, by the the atomic case of the present lemma, the only state in $\mathfrak{M}$ where i is true) iff $\psi \in \Delta_i$ (inductive hypothesis) iff $@_i\psi \in \Delta_i$ (using the fact that $i \in \Delta_i$ together with *introduction* for the left to right direction and *elimination* for the right to left direction) iff $@_i\psi \in u$ (@-agreement).

Completeness

Theorem (Completeness)

Every $\mathbf{K}_h + \textsc{rules}$*-consistent set of formulas in language* $\mathcal{L}$ *is satisfiable in a countable named model. Moreover, if* Π *is a set of pure formulas (in* $\mathcal{L}$*), and* $\mathbf{P}$ *is the normal hybrid logic obtained by adding all the formulas in* Π *as extra axioms to* $\mathbf{K}_h + \textsc{rules}$*, then every* $\mathbf{P}$*-consistent set of sentences is satisfiable in a countable named model based on a frame which validates every formula in* Π*.*

Theorem (Completeness)

Every $\mathbf{K}_h + \textsc{rules}$*-consistent set of formulas in language* $\mathcal{L}$ *is satisfiable in a countable named model. Moreover, if* Π *is a set of pure formulas (in* $\mathcal{L}$*), and* $\mathbf{P}$ *is the normal hybrid logic obtained by adding all the formulas in* Π *as extra axioms to* $\mathbf{K}_h + \textsc{rules}$*, then every* $\mathbf{P}$*-consistent set of sentences is satisfiable in a countable named model based on a frame which validates every formula in* Π*.*

Proof For the first claim, given a $\mathbf{K}_h + \textsc{rules}$-consistent set of formulas Σ, use the Extended Lindenbaum Lemma to expand it to a named and pasted set Σ^+ in a countable language $\mathcal{L}'$. Let $\mathfrak{M} = (W, R, V)$ be the named model yielded by Σ^+. By item (iv) of Lemma 3, because Σ^+ is named, $\Sigma^+ \in W$. By the Truth Lemma, $\mathfrak{M}, \Sigma^+ \Vdash \Sigma$. The model is countable because each state is named by some $\mathcal{L}'$ nominal, and there are only countably many of these.

Completeness Proof continued

So far so good: we've got the minimal logic. What about when we add Pure Axioms

Completeness Proof continued

So far so good: we've got the minimal logic. What about when we add Pure Axioms

For the 'moreover' claim, given a **P**-consistent set of formulas Ξ, use the Extended Lindenbaum Lemma to expand it to a named pasted **P**-MCS Ξ^+. The named model $\mathfrak{M}^\Xi$ that Ξ^+ gives rise to will satisfy Ξ at Ξ^+; but in addition, as every formula in Π belongs to every **P**-MCS, we have that $\mathfrak{M}^\Xi \Vdash \Pi$. Hence, by the Magic Lemma, the frame underlying $\mathfrak{M}^\Xi$ validates Π.

Example

Example

We know that $i \to \neg\Diamond i$ defines irreflexivity and $\Diamond\Diamond i \to \Diamond i$ defines transitivity, hence adding these formulas as axioms to $\mathbf{K}_h + \textsc{rules}$ yields a logic (let us call it **I4**) which is complete with respect to the class of strict preorders.

Example

We know that $i \to \neg\Diamond i$ defines irreflexivity and $\Diamond\Diamond i \to \Diamond i$ defines transitivity, hence adding these formulas as axioms to $\mathbf{K}_h$ + RULES yields a logic (let us call it **I4**) which is complete with respect to the class of strict preorders.

Hence $\Diamond\Diamond p \to \Diamond p$, the ordinary modal transitivity axiom, must be **I4**-provable.

Example

We know that $i \to \neg\Diamond i$ defines irreflexivity and $\Diamond\Diamond i \to \Diamond i$ defines transitivity, hence adding these formulas as axioms to $\mathbf{K}_h$ + RULES yields a logic (let us call it **I4**) which is complete with respect to the class of strict preorders.

Hence $\Diamond\Diamond p \to \Diamond p$, the ordinary modal transitivity axiom, must be **I4**-provable.

Furthermore, as $i \to \neg\Diamond\Diamond i$ is valid on any asymmetric frame, and $i \to \Box(\Diamond i \to i)$ is valid on any antisymmetric frame, these must be **I4**-provable too.

Example

We know that $i \to \neg\Diamond i$ defines irreflexivity and $\Diamond\Diamond i \to \Diamond i$ defines transitivity, hence adding these formulas as axioms to $\mathbf{K}_h + \textsc{rules}$ yields a logic (let us call it **I4**) which is complete with respect to the class of strict preorders.

Hence $\Diamond\Diamond p \to \Diamond p$, the ordinary modal transitivity axiom, must be **I4**-provable.

Furthermore, as $i \to \neg\Diamond\Diamond i$ is valid on any asymmetric frame, and $i \to \Box(\Diamond i \to i)$ is valid on any antisymmetric frame, these must be **I4**-provable too.

Try proving these!

More about our stars (I)

$$(\textsc{name})\ \frac{\vdash j \to \theta}{\vdash \theta} \qquad\qquad (\textsc{paste})\ \frac{\vdash @_i\Diamond j \wedge @_j\phi \to \theta}{\vdash @_i\Diamond\phi \to \theta}$$

- You've seen out two stars before — when we did tableau!
- Name is essentially the way we start a tableau proof. Factoid: in axiomatic proofs, Name is only ever needed once, right at the very end of a proof. See Blackburn and ten Cate, Pure Extensions, Proof Rules, and Hybrid Axiomatics, *Studia Logica*, 2006, 84:277-322.
- And Paste is just the crucial diamond rule lightly disguised!

That is, these rules are natural

More about our stars (II)

$$(\textsc{name})\ \frac{\vdash j \to \theta}{\vdash \theta} \qquad (\textsc{paste})\ \frac{\vdash @_i\Diamond j \wedge @_j\phi \to \theta}{\vdash @_i\Diamond\phi \to \theta}$$

- Stars are needed! We can't prove our general completeness result without using additional rules.
- See Blackburn and ten Cate, Pure Extensions, Proof Rules, and Hybrid Axiomatics, *Studia Logica*, 2006, 84:277-322.

That is, these rules)or variants) are indispensable

Easy to extend acclimatisation to cover $\downarrow$ (I)

- In fact, we can do so simply by adding the following axiom:

$$@_i(\downarrow s.\phi \leftrightarrow \phi[s := i]$$

- Note: this axiom expresses exactly the same idea as yesterday tableaux rules for $\downarrow$.

Easy to extend acclimatisation to cover $\downarrow$ (I)

- In fact, we can do so simply by adding the following axiom:

$$@_i(\downarrow s.\phi \leftrightarrow \phi[s := i]$$

- Note: this axiom expresses exactly the same idea as yesterday tableaux rules for $\downarrow$.

But there is another way. With downarrow in the language, it turn out that we don't need to express the ideas underlying Name and Paste as rules — we have axioms that do the same job . . .

Using $\downarrow$ to eliminate Name and Paste

- $@_i(\downarrow s.\phi \leftrightarrow \phi[s := i]$
- $\downarrow s.(s \to \phi) \to \phi$, provided s does not occur in ϕ. This axiom takes over from the Name rule.
- $@_i\Box \downarrow s.@_i\Diamond s$. This axiom takes over from the Paste rule.

Using $\downarrow$ to eliminate Name and Paste

- $@_i(\downarrow s.\phi \leftrightarrow \phi[s := i]$
- $\downarrow s.(s \rightarrow \phi) \rightarrow \phi$, provided s does not occur in ϕ. This axiom takes over from the Name rule.
- $@_i\Box \downarrow s.@_i\Diamond s$. This axiom takes over from the Paste rule.

Also need: If $\vdash \phi$ then $\vdash \downarrow s\phi$.

For more details, see Blackburn and ten Cate, Pure Extensions, Proof Rules, and Hybrid Axiomatics, *Studia Logica*, 2006, 84:277-322.