Hybrid Logic Tango

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Def: A logic $\mathcal{L}$ has Craig interpolation if the following condition holds. Let φ, ψ be formulas in $\mathcal{L}$ such that

$$\models \varphi \rightarrow \psi.$$

Then there exists a formula θ such that

- $\models \varphi \rightarrow \theta$ and $\vdash \theta \rightarrow \psi$.
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- Intuitively: If $\models \varphi \rightarrow \psi$ then "the implication goes through only in the shared language".

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  $$\models \varphi(\bar{a}, a/b_1) \wedge \varphi(\bar{a}, a/b_2) \rightarrow (b_1 \leftrightarrow b_2).$$

  Then there exists a formula $\theta(\bar{a})$ such that
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- In automated theorem proving: Interpolation has been used to guide the search of intermidiate lemas.

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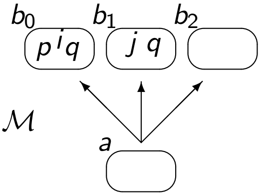
- if there is a unique accessible r-world and
- one of the accessible $\neg r$-worlds is named by the nominal j,
- then the second accessible $\neg r$-world is named $\neg j$.

Proving failure of Interpolation



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Thm: $\mathcal{H}(@, \downarrow)$ has Craig interpolation and Beth definability.

More precisely: Thm: Let K be a class of frames defined by a pure formula. Then the hybrid logic of K over the language $\mathcal{H}(@, \downarrow)$ has Craig interpolation and Beth definability.

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- A similar construction has been used already in [Blackburn and Tzakova, 1999] to prove general completeness results for hybrid logics.

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- Thm: Let K be a class of frames defined by a pure formula. Then the hybrid logic of K over the language $\mathcal{H}(@, \downarrow)$ has Craig interpolation and Beth definability.