Independence friendly logic with classical negation via flattening is a second-order logic with weak dependencies

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Abstract

It is well-known that Independence Friendly (IF) logic is equivalent to existential second-order logic (Σ_1^1) and, therefore, is not closed under classical negation. The Boolean closure of IF sentences, called Extended IF-logic, on the other hand, corresponds to a proper fragment of Δ_2^1. In this article we consider SL($\downarrow$), IF-logic extended with Hodges' flattening operator $\downarrow$, which allows to define a classical negation. Furthermore, this negation, in Hodges' style, may occur also under the scope of IF quantifiers. SL($\downarrow$) contains Extended IF-logic and hence it is at least as expressive as the Boolean closure of Σ_1^1. We prove that SL($\downarrow$) corresponds to a weak syntactic fragment of SO which we show to be strictly contained in Δ_2^1. The separation is derived almost trivially from the fact that Σ_n^1 defines its own truth-predicate. We finally show that SL($\downarrow$) is equivalent to the logic of Henkin quantifiers, which shows, we argue, that Hodges' notion of negation is adequate.

1 Introduction

Independence Friendly logic (IF, for short), introduced by Hintikka and Sandu [1] and became part of Hintikka's foundational programme for mathematics [2], is an extension of first-order logic (FO) where each disjunction and each existential quantifier may be decorated with denotations of universally quantified variables, as in:

$$\forall x \forall y \exists z_{|\forall y} \exists w_{|\forall y} [y \approx z \vee_{|\forall x, \forall y} w \approx y]. \tag{1}$$

The standard interpretation of IF is through a variation of the classical game-theoretical semantics for FO: Eloïse's strategy function for a position of the form $\exists x_{|\forall y, \forall z} \psi$ or $\psi \vee_{|\forall y, \forall z} \chi$, under valuation v, cannot depend on neither $v(y)$ nor $v(z)$. Thus, we say that a sentence φ is *true* in model $\mathcal{A}$ (notation, $\mathcal{A} \models^+ \varphi$) if Eloïse has a winning strategy on the associated game; and that it is *false* (notation, $\mathcal{A} \models^- \varphi$) whenever Abélard has a winning strategy.

Now, the fact that Eloïse's strategy may not take into account all the available information turns the game into one of imperfect information. Thus, certain formula-structure pairs may have a non-determined semantic game; that is, one in which neither of the players has a winning strategy. As an example of non-determinacy, consider this formula:

$$\chi_1 := \forall x \exists y_{|\forall x} x \not\approx y. \tag{2}$$

It is not hard to see that if $\mathcal{A}$ is a model with at least two elements, then $\mathcal{A} \not\models^+ \chi_1$ and $\mathcal{A} \not\models^- \chi_1$. One says that χ_1 is neither *true* nor *false* in $\mathcal{A}$.

In game-theoretical semantics, negation is interpreted as a switch of roles, i.e., Abélard plays on Eloïse's former positions and vice versa. We use $\sim$ to denote this form of negation and we refer to it as *game negation*. For any IF-formula ψ and any model $\mathcal{A}$, $\mathcal{A} \models^+ \psi$ iff $\mathcal{A} \models^- \sim\psi$. (i.e., Eloïse has a winning strategy for ψ on $\mathcal{A}$ iff Abélard has one for $\sim\psi$ on $\mathcal{A}$). However, observe that $\psi \vee \sim\psi$ is not in general a valid IF-formula (e.g., take ψ to be χ_1 in (2)). This means that game negation in IF is not equivalent to *classical negation*, which will be denoted with $\neg$ and is characterized by:

$$\mathcal{A} \models^+ \neg\psi \text{ iff } \mathcal{A} \not\models^+ \psi. \tag{3}$$

Since the expressive power of IF corresponds to that of existential second-order logic (Σ^1_1) [2, 3] and Σ^1_1 is not closed under (classical) negation, it is clear that classical negation cannot be defined in IF.

Classical negation plays an important role in Hintikka's original programme. In [2], he claims that "virtually all of classical mathematics can in principle be done in extended IF first-order logic" (in a way that is ultimately "reducible" to plain IF logic). What he calls "(truth-functionally) extended IF logic" is the closure of the set of IF-sentences with operators $\neg$, $\wedge$ and $\vee$. Clearly, extended IF logic corresponds in expressive power to the Boolean closure of Σ^1_1, which is known to be a proper fragment of Δ^1_2 [4, 5].

Hodges [6] shows that IF logic admits a Tarski-style compositional semantics and then extends his presentation to account also for extended IF. To support classical negation, he introduces the *flattening* operator $\downarrow$, which "restores two-valued logic on sentences" [6, p. 556]. That is, extended IF is obtained, roughly speaking, by considering the formulas where $\downarrow$ only occurs on certain positions (roughly speaking, $\sim\downarrow$ can occur where $\neg$ would occur in extended IF logic, see below). But because $\downarrow$ is given a compositional semantics, the logic where it is allowed to occur anywhere in a formula is well-defined. The natural question to ask is what is the logic one thus obtains, and this is the main topic of this paper.

One might suspect the resulting logic to be extremely expressive: freely combining classical negation with second order existential quantifiers leads to full second-order logic (SO). We will show that this is not the case: IF with unrestricted classical negation, in Hodges' style, corresponds to a rather mild fragment of SO, which is properly contained in Δ^1_2. This will be the subject of Section 4. The separation from Δ^1_2 is based on known results on truth-definitions for the analytical hierarchy [7, 8] that, for the sake of completeness, are presented in Section 6.

Hodges' overall presentation is based on a mild extension of IF, called *slash logic* (SL), in which independence restrictions can occur in any connective (instead of only on $\exists$ and $\vee$). The unique feature of his *compositional* semantics is that the free variables are interpreted using a *set of variable assignments* (called *deals*), instead of just a variable assignment as in usual Tarksi-style semantics for FO. In his terminology, a *trump* for a given game is a non-empty set of deals, V, such that some uniform strategy for Eloïse is winning for every game starting with any $v \in V$. To support classical negation, he extends slash logic with the flattening operator $\downarrow$. If we denote a set of variable assignments with V, its semantics can be given by:

$$\mathcal{A} \models^+ \downarrow\varphi[V] \text{ iff } \mathcal{A} \models^+ \varphi[\{v\}] \quad \text{for all } v \in V; \tag{4}$$

$$\mathcal{A} \models^- \downarrow\varphi[V] \text{ iff } \mathcal{A} \not\models^+ \varphi[\{v\}] \quad \text{for all } v \in V. \tag{5}$$

Then one defines $\neg\varphi$ as $\sim\downarrow\varphi$ and it is easy to verify that when restricted to formulas evaluated under a set composed of a single assignment $\{v\}$ (we omit the braces for readability), negation behaves as expected:

$$\mathcal{A} \models^+ \neg\varphi[v] \text{ iff } \mathcal{A} \models^+ \sim\downarrow\varphi[v] \text{ iff } \mathcal{A} \models^- \downarrow\varphi[v] \text{ iff } \mathcal{A} \not\models^+ \varphi[v]; \tag{6}$$

$$\mathcal{A} \models^- \neg\varphi[v] \text{ iff } \mathcal{A} \models^- \sim\downarrow\varphi[v] \text{ iff } \mathcal{A} \models^+ \downarrow\varphi[v] \text{ iff } \mathcal{A} \models^+ \varphi[v]. \tag{7}$$

It is worth stressing out that the asymmetry in clauses (4) and (5), which in turns reflects in the asymmetry in (6) and (7) is fine. For instance, if in (5) the $\not\models^+$ were replaced by $\models^-$ then one would have that $\neg$ behaves exactly as $\sim$. Observe also that the semantics of $\downarrow$ is biased towards *falsity*: if a sentence φ in SL is neither true nor false then $\downarrow\varphi$ is false. Thus, when working with $\downarrow$, the adequate notion to study is being *true* ($\models^+$) vs. *not true* ($\not\models^+$) instead of being *true* vs. *false*. This is why we will study only the notion $\models^+$ in the context of SL with the operator $\downarrow$.

Hodges' slash logic with flattening (SL($\downarrow$)) admits a more convenient *second-order game* semantics, in which Abélard and Eloïse play what can be regarded as strategy functions for the standard game for SL. This will be the topic of Section 2; for a proof of the equivalence with the original compositional semantics, the reader is referred to [9].

Arguably, it could be possible that the semantics given to the flattening operator only made sense when restricted to sentences. Put in other words, it is not clear a priori that Hodges' characterization of classical negation for IF is the correct one. We investigate this in Section 5; we will see that SL($\downarrow$) coincides with the logic of Henkin quantifiers. The latter can be seen as the closure by (classical) negation of the logic in which only one top-level Henkin quantifier can be used, which is known to be equivalent to IF.

Some of the results contained in the present paper appeared in [10].

2 Syntax and semantics of SL($\downarrow$)

We assume a fixed first-order relational language $\mathcal{L}$, as well as a collection of first-order variables, which we will denote x, y, z, perhaps with subindices. Formulas of SL($\downarrow$), in negation normal form, correspond to the following grammar:

$$\varphi ::= l(x_1, \ldots, x_k) \mid \exists x_{i|\rho}\varphi \mid \forall x_{i|\rho}\varphi \mid {\downarrow}\varphi \mid {\uparrow}\varphi \mid \varphi \vee_{|\rho} \varphi \mid \varphi \wedge_{|\rho} \varphi \tag{8}$$

where ρ denotes a (possibly empty) finite set of variables and $l(x_1, \ldots, x_k)$ is any first-order literal (i.e., an atom or a negated atom). We will typically use $\exists x_i$, $\forall x_i$, $\vee$ and $\wedge$ instead of $\exists x_{i|\emptyset}$, $\forall x_{i|\emptyset}$, $\vee_{|\emptyset}$ and $\wedge_{|\emptyset}$. Since we are working in negation normal form, game negation $\sim$ will be a mapping on formulas satisfying $\sim\forall x_{i|\rho}\varphi = \exists x_{i|\rho}\sim\varphi$; $\sim{\downarrow}\varphi = {\uparrow}\sim\varphi$, etc. Finally, $\neg\varphi$ will be short for $\sim{\downarrow}\varphi$.

$\mathrm{Bv}(\varphi)$ and $\mathrm{Fv}(\varphi)$ denote the sets of bound and free variables of φ, respectively, which are defined as in the classical case with the proviso that variables mentioned in independence restrictions are considered free; e.g., $\mathrm{Fv}(\exists x_{|\rho}\varphi) = (\mathrm{Fv}(\varphi) \setminus \{x\}) \cup \rho$ (see [9] for a formal definition). A *sentence* is a formula with no free variables. A *fresh* variable for a formula is a variable that is not bound nor free for that formula [11, 12]. In order to give a formal account of the semantics of this logic, we need to refer to the *live* variables for a subformula ψ of φ (here we assume that ψ denotes not only a formula but a concrete node in the derivation tree of φ). Intuitively, these are the free variables of φ plus any variable y that would be bound by a quantifier if we substituted ψ by $y \approx y$ in ψ (cf. [13]). Formally, the set $\mathrm{Lv}^{\varphi}(\psi)$ is defined inductively from top down as follows:

1. $\mathrm{Lv}^{\varphi}(\varphi) = \mathrm{Fv}(\varphi)$.
2. If ψ occurs in φ under $\star \in \{\sim, \downarrow, \uparrow\}$, then $\mathrm{Lv}^{\varphi}(\psi) = \mathrm{Lv}^{\varphi}(\star\psi)$.
3. If ψ occurs in φ in the form $\psi \odot \chi$ (resp. $\chi \odot \psi$), with $\odot \in \{\vee_{|\rho}, \wedge_{|\rho}\}$, then $\mathrm{Lv}^{\varphi}(\psi) = \mathrm{Lv}^{\varphi}(\psi \odot \chi)$ (resp., $\mathrm{Lv}^{\varphi}(\psi) = \mathrm{Lv}^{\varphi}(\chi' \odot \psi)$).
4. If ψ occurs in φ under $Qx_{|\rho}$ with $Q \in \{\exists, \forall\}$, then $\mathrm{Lv}^{\varphi}(\psi) = \mathrm{Lv}^{\varphi}(Qx_{|\rho}\psi) \cup \{x\}$.

Remark 1. For the sake of simplicity we will impose a further restriction on formulas: there can be no nested bindings of the same variable (e.g., $\exists x \exists x \varphi$) nor a variable that occurs both free and bound in a formula (e.g., $x \approx y \vee \exists x \varphi$ or $\exists x_{|x}\varphi$). This is called the *regular fragment* of SL($\downarrow$) [12] and it has simpler formal semantics. The results in this paper apply to the whole language under the proviso that *history-preserving valuations* are used instead of standard ones (cf. [9] for details).

We interpret SL($\downarrow$)-formulas using first-order models $\mathcal{A}$ with domain $|\mathcal{A}|$. We use sets of *finite* valuations to account for free variables; the domains of these valuations must be large enough to interpret them all (but they can be larger).

Definition 1. Given φ and $\mathcal{A}$, we say that, V, a set of finite valuations over $\mathcal{A}$, is *suitable for* φ iff there is a finite set $D \supseteq \mathrm{Fv}(\varphi)$ such that $V \subseteq |\mathcal{A}|^D$ and $D \cap \mathrm{Bv}(\varphi) = \emptyset$ (cf. [12]). We say that a finite valuation v over $\mathcal{A}$ is suitable whenever $\{v\}$ is suitable.

We define now the game $\mathsf{G}(\mathcal{A}, \varphi, V)$, where $\mathcal{A}$ is a model and V is a set of finite valuations over $\mathcal{A}$ suitable for φ. As is customary, this game is played between two opponents: Abélard and Eloïse (sometimes called *Falsifier* and *Verifier*). There is also a third agent, called Nature, which acts either as a generator of random choices or as a referee.

The board Game $\mathsf{G}(\mathcal{A}, \varphi, V)$ is played over the syntactic tree of φ. There is, additionally, a set of variables D and a placeholder for a valuation $v : D \to |A|$. Initially, D is such that $V \subseteq |\mathcal{A}|^D$ and v is empty. In the syntactic tree of φ, all the $\exists$, $\vee$ and $\downarrow$-nodes of the tree *belong* to Eloïse; while the $\forall$, $\wedge$ and $\uparrow$-nodes belong to Abélard. Moreover, $\exists$, $\forall$, $\vee$ and $\wedge$-nodes will be (repeatedly) decorated with functions during the game; the first two admit any function $f : |\mathcal{A}|^{D \cup \mathrm{Lv}^{\varphi}(\psi)} \to |\mathcal{A}|$; the last two, only functions $f : |\mathcal{A}|^{D \cup \mathrm{Lv}^{\varphi}(\psi)} \to \{L, R\}$, where ψ stands for the formula that corresponds to the node in question. Initially, the nodes have no decoration.

The turns At any point of the game, the remaining number of turns is bounded by the maximum number of nested occurrences of $\downarrow$-nodes and $\uparrow$-nodes in the game-board.

- *The opening turn.* The first turn is different from the rest. It is composed of two clearly distinguished phases. In the first phase, both players decorate all their nodes with suitable functions. The order in which they tag their nodes is not important as long as they do not get to see their opponent's choices in advance. For simplicity, we will assume they both play simultaneously. In the second phase, Nature picks a valuation from V and puts it in the placeholder v and finally *evaluates* the outcome of the turn, as described below.

- *The subsequent turns.* In all but the first turn, the formula tree is of the form $\downarrow\psi$ or $\uparrow\psi$ (see next). In these turns, both players get to redecorate their nodes, one after the other; Eloïse goes first when the formula tree is of the form $\downarrow\psi$ and Abélard does so on $\uparrow\psi$. Finally, Nature replaces the tree with ψ and proceeds to evaluate.

The recursive evaluation procedure used by Nature is the following:

R1 If the tree has root ψ, of the form $\psi_1 \vee_{|y_1,\ldots,y_k} \psi_2$ or $\psi_1 \wedge_{|y_1,\ldots,y_k} \psi_2$, then ψ must have been decorated with a function $f : |\mathcal{A}|^{D \cup \mathrm{Lv}^{\varphi}(\psi)} \to \{L, R\}$. Nature picks elements $a_1 \ldots a_k$ from $|\mathcal{A}|$ and proceeds to evaluate $f(v[y_1 \mapsto a_1, \ldots, y_k \mapsto a_k])$ —by construction $\{y_1, \ldots y_k\} \subseteq D$. That is, the values the player was not supposed to consider are randomly replaced prior to evaluating the function provided. The tree is then updated with ψ_1, if the result is L, and with ψ_2, otherwise. D and v remain unchanged and evaluation proceeds.

R2 If the tree has root ψ, of the form $\exists x_{|y_1,\ldots,y_k} \psi$ or $\forall x_{|y_1,\ldots,y_k} \psi$, then ψ must have been decorated with a function $f : |\mathcal{A}|^{D \cup \mathrm{Lv}^{\varphi}(\psi)} \to |\mathcal{A}|$. Nature picks $a_1 \ldots a_k$, evaluates $b := f(v[y_1 \mapsto a_1, \ldots, y_k \mapsto a_k])$ and records this choice by replacing D with $D \cup \{x\}$ and v with $v \cup \{x \mapsto b\}$. Finally, the tree is updated with ψ and evaluation proceeds.

R3 If the tree is of the form $\downarrow\psi$ or $\uparrow\psi$, the evaluation ends (and so does the turn).

R4 Finally, if the root of the tree is a literal $l(x_1, \ldots, x_k)$, the game ends. Eloïse is declared the winner if $\mathcal{A} \models_{\mathrm{FO}} l(x_1, \ldots, x_k)[v]$; otherwise, Abélard wins.

Remark 2. Let φ be $\downarrow$- and $\uparrow$-free; then game $\mathsf{G}(\mathcal{A}, \varphi, V)$ consists of only one turn, but the evaluation phase is essentially the usual game for SL (and, mutatis mutandis, for IF), except that Abélard and Eloïse are substituted by the (strategy) functions they already played.

Winning strategies We will not go into a formal description of what a strategy for $\mathsf{G}(\mathcal{A}, \varphi, V)$ is. We simply take it to be a form of oracle that tells the player how to proceed in each turn. As usual, a strategy is said to be *winning* for a player if it guarantees that the he or she will win every instance of the game, regardless the strategy of the opponent and the choices made by Nature.

Definition 2. Let V be a set of finite valuations suitable for φ. We define:

- $\mathcal{A} \models^{+} \varphi[V]$ iff Eloïse has a winning strategy for the game $\mathsf{G}(\mathcal{A}, \varphi, V)$;
- $\mathcal{A} \models^{-} \varphi[V]$ iff Abélard has a winning strategy for the game $\mathsf{G}(\mathcal{A}, \varphi, V)$.

When $V = \{v\}$ we may alternatively write $\mathcal{A} \models^{+} \varphi[v]$ and $\mathcal{A} \models^{-} \varphi[v]$. Also, for a *sentence* φ we may write $\mathcal{A} \models^{+} \varphi$ and $\mathcal{A} \models^{-} \varphi$ meaning $\mathcal{A} \models^{+} \varphi[\varnothing]$ and $\mathcal{A} \models^{-} \varphi[\varnothing]$, respectively, where $\varnothing$ is the empty valuation. If V is a set of finite valuation suitable for φ and W consists of extensions of the valuations in V, from $\mathcal{A} \models^{+} \varphi[W]$ we cannot infer $\mathcal{A} \models^{+} \varphi[V]$. This is due to *signaling*: the value of a variable a player is supposed not to know is available through the value of another one (cf. [14, 15]). It is shown in [9] that the logic of Definition 2 coincides with Hodges' compositional semantics for SL($\downarrow$) described in Section 1.

We will work with two different kind of equivalences.

Definition 3 (Equivalence)**.** We say that φ_1 and φ_2 are *equivalent* (notation: $\varphi_1 \equiv \varphi_2$) whenever $\mathrm{Fv}(\varphi_1) = \mathrm{Fv}(\varphi_2)$, $\mathcal{A} \models^{+} \varphi_1[V] \iff \mathcal{A} \models^{+} \varphi_2[V]$, and $\mathcal{A} \models^{-} \varphi_1[V] \iff \mathcal{A} \models^{-} \varphi_2[V]$, for every $\mathcal{A}$ and every set V suitable for φ_1 and φ_2.

On the other hand, we resort to a coarser notion of equivalence which only considers singletons $V = \{v\}$. We need this in order to compare SL($\downarrow$) with "classical" logics such as second order logic (whose formulas are evaluated in classical valuations).

Definition 4 (Equivalence on classical contexts)**.** We say that φ_1 and φ_2 are *equivalent on classical contexts* (notation: $\varphi_1 \equiv_c \varphi_2$) whenever $\mathrm{Fv}(\varphi_1) = \mathrm{Fv}(\varphi_2)$, $\mathcal{A} \models^{+} \varphi_1[v] \iff \mathcal{A} \models^{+} \varphi_2[v]$ and $\mathcal{A} \models^{-} \varphi_1[v] \iff \mathcal{A} \models^{-} \varphi_2[v]$, for every $\mathcal{A}$ and every valuation v suitable for φ_1 and φ_2.

Observe that Hodges shows that for any *sentence* φ, $\mathcal{A} \models^{+} \varphi$ if and only if the meaning of φ is nonempty. This is the same as saying that the empty valuation belongs to the meaning of φ. Hence, two sentences φ_1 and φ_2 are equivalent if and only if for any structure $\mathcal{A}$, we have $\mathcal{A} \models^{+} \varphi_1$ iff $\mathcal{A} \models^{+} \varphi_2$.

Proposition 1. *Given v, a finite valuation over $\mathcal{A}$ suitable for φ, we have that $\mathcal{A} \models^{+} \varphi[v]$ iff $\mathcal{A} \models^{+} {\downarrow}\varphi[v]$, and that $\mathcal{A} \models^{-} \varphi[v]$ iff $\mathcal{A} \models^{-} {\uparrow}\varphi[v]$.*

Proof. The left-to-right directions hold for all V. For the remaining case, note that Nature's initial choice of a valuation is irrelevant in this case, so if a player has a winning strategy playing first, this same strategy can be used for the case where they play simultaneously. $\square$

To see that Proposition 1 fails when V is not a singleton set, consider $|\mathcal{A}| = \{a, b\}$ and $V = \{\{x \mapsto a\}, \{x \mapsto b\}\}$. It is easy to verify that for $\varphi = \exists y_{|x}[x \approx y]$ we have $\mathcal{A} \models^+ \downarrow\varphi[V]$ (since Eloïse knows the valuation picked by Nature, she can play a constant function for her existential) while $\mathcal{A} \not\models^+ \varphi[V]$.

Remark 3. Nodes may get redecorated during the game but only by its owner, that is fixed. Hence it is equivalent to assume that players decorate only those nodes that are not under nested $\downarrow$ or $\uparrow$. This way, each node gets decorated only once. Moreover, whenever one is interested in whether $\mathcal{A} \models^+ \downarrow\varphi[V]$ holds, it may be convenient to consider an equivalent version of $\mathsf{G}(\mathcal{A}, \downarrow\varphi, V)$ in which Eloïse plays functions and Abélard plays *elements* (until the game reaches a $\uparrow$, where the situation gets reversed). This resembles the perfect-information game for IF given by Väänänen in [16].

Under some assumptions, operators $\downarrow$ and $\uparrow$ turn a formula that may lead to a non-determined game, into one that always leads to a determined one. This suggests the following notion.

Definition 5 (Determined)**.** We say that φ is *determined* whenever, for every model $\mathcal{A}$, and every set V suitable for φ, $\mathcal{A} \not\models^+ \varphi[V]$ iff $\mathcal{A} \models^- \varphi[V]$. All such formulas constitute the *determined fragment of* SL($\downarrow$).

Intuitively, determined formulas are those that have a well-defined truth-value on every structure. One would like that first-order formulas (i.e., those with no independence restrictions) be determined. However this is not the case: the formula $x \approx y$ is not determined when $|\mathcal{A}| = \{a, b\}$ and $V = \{\{x \mapsto a, y \mapsto b\}, \{x \mapsto a, y \mapsto a\}\}$. Here the problem resides in the fact that first-order (as well as any logic in classical context, say second-order) involve single valuations instead of sets of valuations. Furthermore, though $\downarrow$ "restores two-valued logic on sentences", it is not true that it restores a two-valued logic on any first order formula, as $\downarrow(x \approx y)$ is not determined for $\mathcal{A}$ and V defined above.

We restrict Definition 5 in order to guarantee that each first-order formula now behaves as we want, and also to ensure that $\downarrow$ and $\uparrow$ "determine" a formula. The idea is to consider not arbitrary V, but singletons $V = \{v\}$.

Definition 6 (Determined on classical contexts)**.** We say that φ is *determined on classical contexts* (*CC-determined* for short) whenever, for every model $\mathcal{A}$, and every finite valuation v suitable for φ, $\mathcal{A} \not\models^+ \varphi[v]$ iff $\mathcal{A} \models^- \varphi[v]$. All such formulas constitute the *CC-determined fragment of* SL($\downarrow$).

Of course, not every SL($\downarrow$) formula is CC-determined. For instance the formula in (2) is not CC-determined. The following result establishes some sufficient conditions for a formula to be CC-determined:

Proposition 2. *The following hold:*

1. *Every* FO *formula is a CC-determined formula.*
2. *$\downarrow\psi$ and $\uparrow\psi$ are CC-determined formulas.*
3. *If φ and ψ are CC-determined, so are $\varphi \wedge_{|\emptyset} \psi$, $\varphi \vee_{|\emptyset} \psi$, $\exists x_{|\emptyset}\varphi$ and $\forall x_{|\emptyset}\varphi$.*

Proof. For 1, suppose φ is a FO formula. If $\mathcal{A} \models \varphi[v]$ (i.e. φ is true in $\mathcal{A}$ under v with the classic first-order semantics) then Eloïse just plays the winning strategy for the classical game-theoretical semantics for FO, which is a valid winning strategy for the game $\mathsf{G}(\mathcal{A}, \varphi, v)$. Hence if $\mathcal{A} \models \varphi[v]$ then $\mathcal{A} \models^+ \varphi[v]$. Analogously for Abélard: if $\mathcal{A} \not\models \varphi[v]$ then $\mathcal{A} \models^- \varphi[v]$. Since either $\mathcal{A} \models \varphi[v]$ or $\mathcal{A} \not\models \varphi[v]$ holds, φ is CC-determined.

For 2, observe that by (4) and (5) we have that $\mathcal{A} \models^+ \downarrow\varphi[v]$ iff $\mathcal{A} \models^+ \varphi[v]$ iff $\mathcal{A} \not\models^- \downarrow\varphi[v]$. Hence φ is CC-determined.

For 3, observe that on the one hand, $\mathcal{A} \models^+ \varphi \wedge_{|\emptyset} \psi[v]$ iff $\mathcal{A} \models^+ \varphi[v]$, and $\mathcal{A} \models^+ \psi[v]$. On the other hand, $\mathcal{A} \models^- \varphi \wedge_{|\emptyset} \psi[v]$ iff $\mathcal{A} \models^- \varphi[v]$ or $\mathcal{A} \models^- \psi[v]$. Since either $\mathcal{A} \models^+ \varphi[v]$ or $\mathcal{A} \models^- \varphi[v]$ is true, and the same for ψ, then one of $\mathcal{A} \models^+ \varphi \wedge_{|\emptyset} \psi[v]$ or $\mathcal{A} \models^- \varphi \wedge_{|\emptyset} \psi[v]$ must hold. The case for $\varphi \vee_{|\emptyset} \psi$ is analogous.

For $\exists x_{|\emptyset}\varphi$, observe that $\mathcal{A} \models^+ \exists x_{|\emptyset}\varphi[v]$ iff there is $a \in |\mathcal{A}|$ such that $\mathcal{A} \models^+ \varphi[v \cup \{x \mapsto a\}]$, and $\mathcal{A} \models^- \exists x_{|\emptyset}\varphi[v]$ iff for all $a \in |\mathcal{A}|$ we have $\mathcal{A} \models^- \varphi[v \cup \{x \mapsto a\}]$. Since φ is CC-determined, one of these two must hold. The case for $\forall x_{|\emptyset}\varphi$ is analogous. □

As was mentioned in the introduction, the semantics of $\downarrow$ is biased towards *falsity*: if $\mathcal{A} \not\models^+ \varphi[\varnothing]$ and $\mathcal{A} \not\models^- \varphi[\varnothing]$ then $\mathcal{A} \models^- \downarrow\varphi$. Observe that this is a straightforward consequence of Proposition 1 and item 2 of Proposition 2.

3 Normal forms for SL($\downarrow$)

Normal forms in the context of SL were initially investigated in [13]. Later, Janssen [14] observed some anomalies which cast doubt on the correctness of these results. However, it was shown in [12], [11] and [9] that only the formal apparatus employed in [13] was defective, and not the results per se.

In this section we revisit the prenex normal form results of [13] and extend them to account for $\downarrow$ and $\uparrow$. For this, bound variables will be tacitly renamed when necessary[1] and the following formula manipulation tools will be employed.

Definition 7. Let $x_1 \dots x_n$ be variables not occurring in φ; we denote with $\varphi_{|x_1 \dots x_n}$ the formula obtained by adding $x_1 \dots x_n$ as restrictions to every quantifier, every conjunction and every disjunction in φ. Also, we write φ^c for the formula obtained by replacing all independence restrictions in φ by $\emptyset$.

[1] While this assumption was considered problematic in the context of [13], it is safe here since we are using regular formulas. Moreover, this can also be assumed for arbitrary formulas under an adequate formalization (cf. Remark 1).

Notice that φ^c is essentially a FO formula. As is observed in [13], independence restrictions on Boolean connectives can be removed by introducing additional quantifications. It is not hard to extend this result to SL(↓). In what follows, we shall use, for emphasis, $\vee_{|\emptyset}$ and $\wedge_{|\emptyset}$ instead of $\vee$ and $\wedge$, etc.

Theorem 3. *For every φ, there exists a φ' such that $\varphi \equiv \varphi'$ and every disjunction (resp. conjunction) in φ' is of the form $\psi_1 \vee_{|\emptyset} \psi_2$ (resp. $\psi_1 \wedge_{|\emptyset} \psi_2$).*

Proof. When restricted to models with at least two elements, a simple inductive argument gives us the desired formula. The important step is that, given a formula $\psi := \psi_1 \vee_{|x_1 \dots x_k} \psi_2$ and given y_1, y_2 fresh for ψ, we can define

$$\psi^* := \exists y_{1|x_1 \dots x_k} \exists y_{2|x_1 \dots x_k} [(y_1 \approx y_2 \wedge \psi_{1|y_1,y_2}) \vee (y_1 \not\approx y_2 \wedge \psi_{2|y_1,y_2})]. \tag{9}$$

Fact 4. On models $\mathcal{A}$ with at least two elements*, we have $\mathcal{A} \models^+ \psi[V] \iff \mathcal{A} \models^+ \psi^*[V]$ and $\mathcal{A} \models^- \psi[V] \iff \mathcal{A} \models^- \psi^*[V]$.*

Proof. Assume $\mathcal{A} \models^+ \psi[V]$. We transform Eloïse's winning strategy on $\mathsf{G}(\mathcal{A}, \psi, V)$ into a winning strategy of Eloïse for $\mathsf{G}(\mathcal{A}, \psi^*, V)$. Suppose Eloïse plays a $\{L, R\}$-valued function f for the outermost node $\vee_{|x_1 \dots x_k}$ of ψ and let $a, b \in |\mathcal{A}|$ be two distinct elements. Then Eloïse plays the following $|\mathcal{A}|$-valued functions g_1 and g_2 for the outermost nodes $\exists y_{1|x_1 \dots x_k}$ and $\exists y_{2|x_1 \dots x_k}$ of ψ^* respectively, and an $\{L, R\}$-valued function h for the outermost node $\vee$ of ψ^*: if $f(v) = L$ then $g_1(v) = g_2(v) = a$ and $h(v) = L$; if $f(v) = R$ then $g_1(v) = a$, $g_2(v) = b$ and $h(v) = R$. The rest of Eloïse's strategy in $\mathsf{G}(\mathcal{A}, \psi^*, V)$ is the one she has in $\mathsf{G}(\mathcal{A}, \psi, V)$. Observe that $g_1(v)$ and $g_2(v)$ can be determined independently of the values of $x_1 \dots x_k$ because $f(v)$ is determined in that way. It is not hard to check that these definitions of g_1, g_2 and h are winning for Eloïse in $\mathsf{G}(\mathcal{A}, \psi^*, V)$.

Assume now that $\mathcal{A} \models^+ \psi^*[V]$. We transform Eloïse's winning strategy on $\mathsf{G}(\mathcal{A}, \psi^*, V)$ into a winning strategy of Eloïse for $\mathsf{G}(\mathcal{A}, \psi, V)$. Suppose Eloïse plays $|\mathcal{A}|$-valued functions g_1 and g_2 for the outermost nodes $\exists y_{1|x_1 \dots x_k}$ and $\exists y_{2|x_1 \dots x_k}$ of ψ^* respectively. Then we define the function f for the outermost node $\vee_{|x_1 \dots x_k}$ of ψ as follows: if $g_1(v) = g_2(v)$ then $f(v) = L$; otherwise $f(v) = R$. Since the value of $g_1(v)$ and $g_2(v)$ can be determined independently of $x_1 \dots x_k$ then so is the determination of the value for $f(v)$. One can check that this definition of f (together with the rest of Eloïse's strategy for $\mathsf{G}(\mathcal{A}, \psi^*, V)$) constitutes a winning strategy for Eloïse in $\mathsf{G}(\mathcal{A}, \psi, V)$.

It is straightforward to see that $\mathcal{A} \models^- \psi[V]$ iff $\mathcal{A} \models^- \psi^*[V]$. □

By successively applying this truth-preserving transformation in a top-down manner, one can obtain, for any given φ, a formula $\tilde{\varphi}$ that is equivalent on models with at least two elements.

On models with exactly one element, restrictions are meaningless. Therefore, for any given φ we can define the equivalent formula:

$$\varphi' := (\forall x \forall y [x \approx y] \wedge \varphi^c) \vee (\exists x \exists y [x \not\approx y] \wedge \tilde{\varphi}). \tag{10}$$

□

Formula (9) in the above proof was taken from [13], except that we have added independences on y_1 and y_2 to ψ_1 and ψ_2. This prevents undesired signaling [14, 15, 12, 9] and it was most probably an involuntary omission in [13]. Also, since we are considering only suitable valuations, the following result in [13] is now true.

Lemma 5. *If x does not occur in ψ, then the following hold:*

1. $\exists x_{|\rho}[\varphi] \vee_{|\emptyset} \psi \equiv \exists x_{|\rho}[\varphi \vee_{|\emptyset} \psi_{|x}]$.
2. $\exists x_{|\rho}[\varphi] \wedge_{|\emptyset} \psi \equiv \exists x_{|\rho}[\varphi \wedge_{|\emptyset} \psi_{|x}]$.
3. $\forall x_{|\rho}[\varphi] \vee_{|\emptyset} \psi \equiv \forall x_{|\rho}[\varphi \vee_{|\emptyset} \psi_{|x}]$.
4. $\forall x_{|\rho}[\varphi] \wedge_{|\emptyset} \psi \equiv \forall x_{|\rho}[\varphi \wedge_{|\emptyset} \psi_{|x}]$.

The above result is a basic building block for a proof of a prenex normal form theorem. In the case of SL($\downarrow$), we also need to show how to extract $\downarrow$ and $\uparrow$ from arbitrary formulas.

Lemma 6. *The following hold:*

1. *If ψ is CC-determined, then* $\downarrow\psi \equiv_c \uparrow\psi \equiv_c \psi$.
2. $\downarrow(\varphi \wedge_{|\emptyset} \psi) \equiv_c \downarrow\varphi \wedge_{|\emptyset} \downarrow\psi$ *and* $\uparrow(\varphi \wedge_{|\emptyset} \psi) \equiv_c \uparrow\varphi \wedge_{|\emptyset} \uparrow\psi$.
3. $\downarrow(\varphi \vee_{|\emptyset} \psi) \equiv_c \downarrow\varphi \vee_{|\emptyset} \downarrow\psi$ *and* $\uparrow(\varphi \vee_{|\emptyset} \psi) \equiv_c \uparrow\varphi \vee_{|\emptyset} \uparrow\psi$.

Proof. Item 1 follows straightforwardly from Proposition 1.

For the first equivalence of item 2, suppose that $\mathcal{A} \models^+ \downarrow(\varphi \wedge_{|\emptyset} \psi)[v]$; this means that Eloïse has a way of decorating both φ and ψ that guarantees she wins each game. Therefore, we have $\mathcal{A} \models^+ \downarrow\varphi[v]$ and $\mathcal{A} \models^+ \downarrow\psi[v]$ which implies $\mathcal{A} \models^+ (\downarrow\varphi \wedge_{|\emptyset} \downarrow\psi)[v]$. The right to left direction is analogous, and one thus establishes that $\mathcal{A} \models^+ \downarrow(\varphi \wedge_{|\emptyset} \psi)[v]$ iff $\mathcal{A} \models^+ (\downarrow\varphi \wedge_{|\emptyset} \downarrow\psi)[v]$. Moreover, since $\downarrow(\varphi \wedge_{|\emptyset} \psi)$ and $(\downarrow\varphi \wedge_{|\emptyset} \downarrow\psi)$ are CC-determined formulas (Proposition 2), this implies $\mathcal{A} \models^- \downarrow(\varphi \wedge_{|\emptyset} \psi)[v]$ iff $\mathcal{A} \models^- (\downarrow\varphi \wedge_{|\emptyset} \downarrow\psi)[v]$.

For the first equivalence of item 3, suppose that $\mathcal{A} \models^+ \downarrow(\varphi \vee_{|\emptyset} \psi)[v]$; this means that Eloïse has a way of decorating either φ or ψ (or both) that guarantees she wins the corresponding game. Therefore, we have $\mathcal{A} \models^+ \downarrow\varphi[v]$ or $\mathcal{A} \models^+ \downarrow\psi[v]$ which implies $\mathcal{A} \models^+ (\downarrow\varphi \vee_{|\emptyset} \downarrow\psi)[v]$. The right to left direction is analogous, and one thus establishes that $\mathcal{A} \models^+ \downarrow(\varphi \vee_{|\emptyset} \psi)[v]$ iff $\mathcal{A} \models^+ (\downarrow\varphi \vee_{|\emptyset} \downarrow\psi)[v]$. Moreover, since $\downarrow(\varphi \vee_{|\emptyset} \psi)$ and $(\downarrow\varphi \vee_{|\emptyset} \downarrow\psi)$ are CC-determined formulas (Proposition 2), this implies $\mathcal{A} \models^- \downarrow(\varphi \vee_{|\emptyset} \psi)[v]$ iff $\mathcal{A} \models^- (\downarrow\varphi \vee_{|\emptyset} \downarrow\psi)[v]$.

The second equivalences of items 2 and 3 are dual of the first equivalences of items 3 and 2 respectively. □

Definition 8. A SL($\downarrow$)-formula is said to be in prenex normal form if it is of the form $Q_0^* \updownarrow_1 Q_1^* \updownarrow_2 \dots Q_{n-1}^* \updownarrow_{n-1} Q_n^* \varphi$ with $n \geq 0$, where each Q_i^* is a (perhaps empty) sequence of quantifiers, $\updownarrow_i \in \{\downarrow, \uparrow\}$ and φ contains only literals, $\wedge_{|\emptyset}$ and $\vee_{|\emptyset}$.

Theorem 7. *For every* SL($\downarrow$)*-formula φ, there exists a φ^* in prenex normal form with* $\varphi \equiv_c \varphi^*$.

Proof. By Theorem 3 we can obtain a φ' such that $\varphi' \equiv_c \varphi$ and no Boolean connective in it contains independences. We proceed now by induction on φ'. If φ' is a literal, $\varphi^* = \varphi'$. If $\varphi' = \exists x_{|y_1 \dots y_k} \psi$, we have $\varphi^* = \exists x_{|y_1 \dots y_k} \psi^*$ and the cases for $\varphi' = \forall x_{|y_1 \dots y_k} \psi$, $\varphi' = \downarrow\psi$ and $\varphi' = \uparrow\psi$ are analogous. We analyze now the case for $\varphi' = \psi \vee \chi$; the one for $\varphi' = \psi \wedge \chi$ is symmetrical.

We need to show that there exists a $\varphi^* \equiv_c (\psi^* \vee \chi^*)$, in prenex normal form. We do it by induction on the sum of the lengths of the prefixes of ψ^* and χ^*. The base case is trivial; for the inductive case we show that one can always "extract" the outermost operator of either ψ^* or χ^*.

The first thing to note is that if $\psi^* = Qx_{|y_1 \dots y_k} \psi'$ ($Q \in \{\forall, \exists\}$), then using Lemma 5 (renaming variables, if necessary) we have $\varphi^* := Qx_{|y_1 \dots y_k} (\psi' \vee \chi^*)^*$ and the same applies to the case $\chi^* = Qx_{|y_1 \dots y_k} \chi'$. So suppose now that neither ψ^* nor χ^* has a quantifier as outermost operator. In that case, they start with one of $\downarrow$ or $\uparrow$, or they contain only $\wedge_{|\emptyset}$, $\vee_{|\emptyset}$ and literals. In either case, they are both CC-determined and at least one of them starts with $\downarrow$ or $\uparrow$ (or we would be in the base case). If we assume that $\psi^* = \downarrow\psi'$, using Lemma 6 repeatedly, we have $(\downarrow\psi' \vee \chi^*) \equiv_c (\downarrow\psi' \vee \downarrow\chi^*) \equiv_c \downarrow(\psi' \vee \chi^*)$, and we can apply the inductive hypothesis. The remaining cases are analogous. □

Observe that in the proof above, the formula φ^* obtained is (strongly) equivalent to the given φ if there are no occurrences of $\downarrow$ nor $\uparrow$ in φ (i.e., $\varphi \in$ SL). Moreover, in that case, no $\downarrow$ nor $\uparrow$ are introduced in the resulting φ^*. Hence, we obtain the following result (cf. Corollary 10.3 in [12]):

Corollary 8. *For every* SL-formula φ, there exists a φ^* in prenex normal form with $\varphi \equiv \varphi^*$

4 Weak dependencies in second-order logic

It is not hard to encode in a SO-formula the game semantics of a CC-determined formula of SL($\downarrow$): quantification over Skolem functions accounts for the functions that can be played by a player while first-order quantification is used for the rival's moves (cf. Remark 3). This will be shown in detail in the proof of Theorem 16 (item 1), but we can now anticipate an interesting feature of this translation: if φ is a formula obtained from it and $\exists f \psi$ is a second-order quantification that occurs inside φ, then although f formally *depends* on any previously quantified function g, in practice it only depends on a finite number of values of such g. This motivates the fragment of SO we are about to introduce which, moreover, will be shown to coincide with SL($\downarrow$) (with respect to $\models^+$).

Assumption 9. In what follows we assume, without loss of generality, that if a variable occurs free in a SO-formula, it does not also appear bound. We reserve letters f, g and h (probably with subindices) to denote second-order functional variables; arities will be left implicit. We identify first-order variables with 0-ary second-order variables; letters x, y and z (with subindices) are to be interpreted always as 0-ary functions (f, g, etc. could be 0-ary too, unless stated). We also assume, as is customary, that only FO terms occur in

SO-formulas (the occurrence of a proper SO terms as in the formula $f \approx g$ for f, g unary may be replaced by $\forall x[f(x) \approx g(x)]$).

Definition 10. We say that an occurrence of the functional symbol f is *strongly free* in a SO-formula φ whenever f is free in φ and, if the occurrence in question is of the form $f(\ldots g(\ldots)\ldots)$, then the occurrence of g is strongly free in φ too. We say that f is strongly free in φ if all its occurrences are strongly free.

Example 1. Variable g is strongly free in $\exists x[f(x) \approx g(y, g(z, y))]$, while f is not (for x is not, either). Any free first-order variable is also strongly free.

Lemma 9. *Let φ be a* SO*-formula and let $g_1 \ldots g_k$ be strongly free in φ. Moreover, let v_1 and v_2 be interpretations of functional variables in $\mathcal{A}$ such that (i) $v_1(f) = v_2(f)$ for every $f \in \mathrm{Fv}(\varphi) \setminus \{g_1, \ldots, g_k\}$, and (ii) for every $g_i(t_1, \ldots, t_m)$ occurring in φ, $v_1(g_i(t_1, \ldots, t_m)) = v_2(g_i(t_1, \ldots, t_m))$. Then, $\mathcal{A} \models_{\mathrm{SO}} \varphi[v_1]$ iff $\mathcal{A} \models_{\mathrm{SO}} \varphi[v_2]$.*

Proof. First we analyze a condition over terms:

Fact 10. *If φ is quantifier-free and t is a term occurring in φ, $v_1(t) = v_2(t)$.*

Proof. By induction on the complexity of the term t. If t is a constant symbol it is straightforward. If t is of the form $h(t_1, \ldots, t_m)$ (or a first-order variable in case $m = 0$) then by inductive hypothesis for all $i = 1 \ldots m$ we have $v_1(t_i) = v_2(t_i)$. In case $h \notin \{g_1, \ldots, g_k\}$, by clause (i) we have $v_1(h) = v_2(h)$ and therefore $v_1(t) = v_2(t)$. In case $h = g_i$ for some i then by clause (ii), we conclude $v_1(t) = v_2(t)$. □

We now show the statement of the lemma by induction on the complexity of φ. Suppose φ is of the form $P(t_1, \ldots, t_m)$, where P is an m-ary relation symbol and $t_1, \ldots, t_m$ are terms. By Fact 10, $v_1(t_i) = v_2(t_i)$ and then $\mathcal{A} \models_{\mathrm{SO}} \varphi[v_1]$ iff $\mathcal{A} \models_{\mathrm{SO}} \varphi[v_2]$. The Boolean cases for φ are straightforward. Finally, suppose φ is of the form $\exists f \psi$. Observe that if $g_1 \ldots g_k$ are strongly free in φ, then $f \notin \{g_1, \ldots, g_k\}$; furthermore, $g_1, \ldots, g_k$ are also strongly free in ψ. Let v_1 and v_2 be interpretations of the variables satisfying the hypothesis and suppose that $\mathcal{A} \models_{\mathrm{SO}} \exists f \psi[v_1]$. Then, there exists $\tilde{f}$ such that $\mathcal{A} \models_{\mathrm{SO}} \psi[v_1 \cup \{f \mapsto \tilde{f}\}]$. Thus, by inductive hypothesis $\mathcal{A} \models_{\mathrm{SO}} \psi[v_2 \cup \{f \mapsto \tilde{f}\}]$. Hence $\mathcal{A} \models_{\mathrm{SO}} \varphi[v_2]$.

□

It is well-known that $\forall x_1 \ldots x_n \exists f \varphi$ is equivalent to $\exists \tilde{f} \forall x_1 \ldots x_n \tilde{\varphi}$, where $\tilde{\varphi}$ is obtained by replacing every occurrence of a term of the form $f(t_1, \ldots, t_k)$ in φ by $\tilde{f}(t_1, \ldots, t_k, x_1, \ldots, x_n)$. The following is a generalization of this idea to strongly free second-order variables used instead of first-order ones.

Theorem 11. *Let $g_1 \ldots g_n$ be strongly free in φ and let h, free in φ, be such that $g_i(\ldots h(\ldots)\ldots)$ does not occur in φ. Then, for every $f_1 \ldots f_m$ free in φ, there exists $\tilde{\varphi}$ such that $g_1 \ldots g_n$ are strongly free in $\tilde{\varphi}$; $f_1 \ldots f_m$ are free in $\tilde{\varphi}$ and $\forall g_1 \ldots \forall g_n \exists h \exists f_1 \ldots \exists f_m \varphi \equiv \exists \tilde{h} \forall g_1 \ldots \forall g_n \exists f_1 \ldots \exists f_m \tilde{\varphi}$.*

Proof. The idea is to move 'h to the front' using the fact that h does not depend on all the values of g_i, but only on finitely many of them.

Let T be the set of terms of the form $g_i(t_1, \dots t_{p_i})$, $1 \leq i \leq n$, occurring in φ. Since $g_1, \dots, g_n$ are strongly free in φ, for each term $g_i(t_1, \dots t_{p_i}) \in T$ and for each $j \in \{1, \dots, p_i\}$ we have that t_j is a term built from constant and function symbols of the language, from variables occurring free in $\forall g_1 \dots \forall g_n \exists h \exists f_1 \dots \exists f_m \varphi$, and from $f_1, \dots, f_m$ —observe that the hypothesis precludes h to occur in t_j. Hence h only depends on the terms in T. Suppose $T = \{s_1, \dots, s_l\}$, suppose h has arity k, and consider $\tilde{h}$ of arity $k + l$. Define $\tilde{\varphi}$ as the result of replacing every occurrence of $h(t_1, \dots, t_k)$ in φ by $\tilde{h}(\tilde{t_1}, \dots, \tilde{t_k}, s_1, \dots, s_l)$, where $\tilde{t_i}$ is the result of the recursive replacement of h by $\tilde{h}$ in t_i, for $i = 1, \dots, k$. Since $g_i(\dots h(\dots) \dots)$ does not occur in φ, no occurrence of h is left in $\tilde{\varphi}$. One can see that $\mathcal{A} \models_{\mathrm{SO}} \forall g_1 \dots \forall g_n \exists h \exists f_1 \dots \exists f_m \varphi[v]$ if and only if $\mathcal{A} \models_{\mathrm{SO}} \exists \tilde{h} \forall g_1 \dots \forall g_n \exists f_1 \dots \exists f_m \tilde{\varphi}[v]$. $\square$

In a way, what Theorem 11 says is that a strongly free second-order variable corresponds, in terms of *information*, to a finite number of first-order terms. Quantification over strongly free second-order variables introduces only "weak" dependencies between them. This is formalized in the following definition.

Definition 11. We say that a SO-formula in prenex normal form has *weak dependencies* if in every subformula of the form $\forall g_1 \dots \forall g_n \exists f_1 \dots \exists f_m \varphi$ (with $\varphi \neq \exists h \psi$) or $\exists g_1 \dots \exists g_n \forall f_1 \dots \forall f_m \varphi$ ($\varphi \neq \forall h \psi$), $g_1 \dots g_n$ are strongly free in φ. This notion is extended to an arbitrary formula φ requiring that the prenex normal forms induced by the branches of the derivation tree of the formula have weak dependencies. We use SO^w to denote the fragment of SO-formulas with weak dependencies.

It is immediate that SO^w is closed under Boolean operations. Moreover, it is not hard (though perhaps rather tedious) to verify that it is also closed under some standard transformations:

Proposition 12. *Every* SO^w*-formula is equivalent to a* SO^w*-formula with only* shallow terms *(i.e., if* $g(t_1, \dots, t_n)$ *occurs in the formula, all the* t_i *are first-order variables) and the same number of quantifier alternations. Every* SO^w*-formula is equivalent to a* SO^w*-formula in prenex normal form, with the same number of quantifier alternations and containing no new terms. Both transformations are primitive recursive.*

Proof. Let φ be a SO^w-formula. For the first statement, suppose that for some relation symbol P, the atomic formula $P(\dots g(t_1, \dots, t_n) \dots)$ occurs in φ, where some t_i is not a first order variable. Then define φ' as the replacement of $P(\dots g(t_1, \dots, t_n) \dots)$ in φ by

$$\exists x_1 \dots \exists x_n \left(\bigwedge_{1 \leq i \leq n} x_i \approx t_i \right) \wedge P(\dots g(x_1, \dots, x_n) \dots),$$

where $x_1 \dots x_n$ are fresh variables, or by

$$\forall x_1 \dots \forall x_n \left(\bigwedge_{1 \leq i \leq n} x_i \approx t_i \right) \rightarrow P(\dots g(x_1, \dots, x_n) \dots)$$

if the first replacement increases the number of quantifier alternations in the resulting formula. Repeating this process for every atomic subformula of the form $P(\dots g(t_1,\dots,t_n)\dots)$ where some t_i is not a first order variable, we obtain the desired equivalent SO^w-formula.

The second statement is a straightforward consequence of the following facts, since we only need to show that the quantifiers can be moved step by step to the front of the formula. Firstly, whenever f is not free in ψ then $(Qf\ \varphi)\star\psi \equiv Qf\ (\varphi\star\psi)$, if $Q\in\{\forall,\exists\}$ and $\star\in\{\vee,\wedge\}$. Secondly, $(Qf\ \varphi)\star\psi$ is a SO^w-formula if and only if $Qf\ (\varphi\star\psi)$ is so. Finally, an analogous rule for negation holds.

It is routine to check that these two transformations are primitive recursive. □

The natural question now is what is the expressive power of this weak fragment. We provide both upper and lower bounds.

Proposition 13. *Every formula in the Boolean closure of Σ^1_1 has an equivalent formula in SO^w.*

Proof. Take any $\varphi\in\Sigma^1_1$; φ can be rewritten as $\exists f_1\dots\exists f_n\forall x_1\dots\forall x_n\varphi'$ with φ' quantifier-free which is trivially a formula in SO^w. Moreover, recall that SO^w is closed under Boolean operations. □

Proposition 14. *There are primitive recursive translations that map any formula in SO^w to equivalent Σ^1_2- and Π^1_2-formulas. Hence SO^w is contained in Δ^1_2.*

Proof. We first fix some notation. Suppose

$$\psi = \exists\vec{h}\forall\vec{g}\exists\vec{f_1}\forall\vec{f_2}\exists\vec{f_3}\dots Q_k\vec{f_k}\ \rho,$$

where ρ is first order, $Q_k=\forall$ if k is odd and $Q_k=\exists$ otherwise, $\exists\vec{h}$ $[\forall\vec{g}]$ is a possibly empty list of existential [universal] quantifiers, and $\forall\vec{f}_{2n+1}$ $[\exists\vec{f}_{2n+2}]$ is a nonempty list of universal [existential] quantifiers, and not all symbols in $\vec{f_k}$ are 0-ary. We say that $Q_i\vec{f_i}$ is the i-th misplaced block. Clearly φ has no misplaced blocks if and only if ψ is Σ^1_2.

We now turn to the proof. Being SO^w closed under negations, it suffices to show a primitive recursive translation from SO^w to Σ^1_2. We actually show that the quantifiers in the prenex normal form of a SO^w-formula with shallow terms (cf. Proposition 12) can be reordered, one at a time, in a top-down manner, leading to a Σ^1_2-formula. That this is a primitive recursive procedure will be immediate.

Suppose, then, that $\varphi\in\mathrm{SO}^w$ is in prenex normal form, it only has shallow terms and it is not in Σ^1_2-form. We convert φ into a Σ^1_2-formula by induction on $\ell(\varphi)$, the sum of the lengths of each misplaced block of φ. For the base case, the transformation is just the identity, as $\ell(\varphi)=0$ implies that φ is a Σ^1_2-formula. For the inductive step, suppose that φ is not in Σ^1_2 form, i.e. $\ell(\varphi)>0$. Consider the least block that is misplaced. There are two possibilities.

1. *Not all the symbols of the first misplaced block are 0-ary.* Suppose

$$\varphi = \exists h_1\dots\exists h_l\forall g_1\dots\forall g_n\exists x_1\dots\exists x_k\exists h\exists f_1\dots\exists f_m\psi,$$

where $k, l, m \geq 0$, $n \geq 1$, $x_1 \dots x_k$ of zero arity and h of *non-zero arity*. Since we assumed φ to have only shallow terms, $g_i(\dots h(\dots)\dots)$ does not occur in ψ. By Theorem 11 we can relocate the misplaced $\exists h$ and obtain the equivalent formula

$$\varphi' = \exists h_1 \dots \exists h_l \exists \tilde{h} \forall g_1 \dots \forall g_n \exists x_1 \dots \exists x_k \exists f_1 \dots \exists f_m \tilde{\psi}.$$

Observe that if $k + m > 0$ the only misplaced block that changes from φ to φ' is the first one (all the others remain the same). In this case the length of the first misplaced block in φ has length $k + m + 1$, while the length of the first misplaced block in φ' has length $k + m$. Hence $\ell(\varphi') < \ell(\varphi)$. In case $k + m = 0$ then φ' has one less misplaced block than φ, but in this case it is also true that $\ell(\varphi') < \ell(\varphi)$. By inductive hypotheses, we can transform φ' into a Σ^1_2-formula.

2. *All the symbols of the first misplaced block are 0-ary.* Here it can be the case that $g_i(\dots x_j \dots)$ occurs in ψ. Therefore, suppose

$$\varphi = \exists h_1 \dots \exists h_l \forall g_1 \dots \forall g_n \exists x_1 \dots \exists x_k \forall f_1 \dots \forall f_m \psi,$$

where $l \geq 0, k, m, n \geq 1$, the variables $x_1, \dots, x_k$ are all first-order, and ψ does not start with $\forall$. In this case we relocate the quantifiers corresponding to the second misplaced block. Using Theorem 11 repeatedly (recall that $x_1 \dots x_k$ are strongly free in ψ), we obtain the equivalent

$$\varphi' = \exists h_1 \dots \exists h_l \forall g_1 \dots \forall g_n \forall \tilde{f}_1 \dots \forall \tilde{f}_m \exists x_1 \dots \exists x_k \tilde{\psi}.$$

Observe that φ' is a SO^w-formula, since $\forall g_1 \dots \forall g_n \forall \tilde{f}_1 \dots \forall \tilde{f}_m \exists x_1 \dots \exists x_k \tilde{\psi}$ is one too. As one can see all the misplaced blocks of φ' are also misplaced blocks of φ, but φ' has one less misplaced block than φ, namely $\forall f_1 \dots \forall f_m$. This implies that $\ell(\varphi') < \ell(\varphi)$ and by inductive hypothesis φ can be transformed into an equivalent Σ^1_2-formula.

The proof straightforwardly induces a primitive recursive procedure for converting φ into a Σ^1_2-formula. □

It will be shown in Section 6 that any logic that can be translated both to Σ^1_n and to Π^1_n ($n > 1$) in a primitive recursive way (in fact, in an arithmetical way) has a truth-predicate in Δ^1_n (cf. Theorem 25). Hence Proposition 14 implies that the truth-predicate of SO^w lies in Δ^1_2. By Tarski's Undefinability Theorem such predicate cannot be expressed in SO^w, which gives us the following:

Corollary 15. SO^w *is strictly contained in* Δ^1_2.

The remaining of this section will be devoted to proving that SO^w coincides in expressive power with $\mathrm{SL}(\downarrow)$. Of course, this transfers the expressiveness bounds of SO^w to $\mathrm{SL}(\downarrow)$.

Definition 12. Let $\varphi \in \mathrm{SL}(\downarrow)$ and $\psi \in \mathrm{SO}^w$. We say that φ and ψ are equivalent, denoted $\varphi \doteq \psi$, whenever for every structure $\mathcal{A}$ and every suitable valuation v, $\mathcal{A} \models^+ \varphi[v]$ iff $\mathcal{A} \models_{\mathrm{SO}} \psi[v]$.

Theorem 16. *The following hold:*

1. *For every $\varphi \in \mathrm{SL}(\downarrow)$, there is a $\varphi^* \in \mathrm{SO}^w$ such that $\varphi \doteq \varphi^*$.*

2. *For every $\psi \in \mathrm{SO}^w$, there is a $\psi^\star \in \mathrm{SL}(\downarrow)$ such that $\psi^\star \doteq \psi$.*

For the proof of item 1 of Theorem 16, we will use an *Skolemization* to show that the existence of a winning strategy for Eloïse in a game $\mathsf{G}(\varphi, \mathcal{A}, \{v\})$ for a φ in prenex normal form can be expressed as SO^w-formula. We first motivate this sort of Skolem form with a short example; so let ψ be quantifier-free, with variables among $\{x_1, x_2, x_3, y_1, y_2, y_3\}$ and consider:

$$\chi_2 := \downarrow\forall y_1 \forall y_2 \exists x_{1|y_2} \uparrow \exists x_{2|y_2} \exists x_3 \forall y_{3|x_3} \psi \tag{11}$$

Assume Eloïse has a winning strategy for $\mathsf{G}(\mathcal{A}, \chi_2, \{v\})$. Using the simplification of Remark 3 this is the case if and only if $\mathcal{A} \models_{\mathrm{SO}} \chi_2'$, where

$$\chi_2' := \exists f \forall y_1 \forall y_2 \forall z_1 \forall g \exists x_2 \exists x_3 \exists z_2 [\psi\sigma_1] \tag{12}$$

and $\sigma_1 = \{x_1 \mapsto f(y_1, z_1), y_3 \mapsto g(y_1, y_2, f(y_1, z_1), x_2, z_2)\}$ is a substitution of variables by terms[2]. Notice that z_1 and z_2 represent the random choices made by Nature during the evaluation phases; e.g., $f(y_1, z_1)$ expresses that Nature replaced the value of y_2 by a randomly picked z_1 when evaluating x_1. Since z_1 and z_2 do not occur in ψ and y_1 and y_2 occur universally quantified, just as g, we have that χ_2' is equivalent to χ_2'', where

$$\chi_2'' := \exists f \forall y_1 \forall y_2 \forall g \exists x_2 \exists x_3 [\psi\sigma_2] \tag{13}$$

and $\sigma_2 = \{x_1 \mapsto f(y_1), y_3 \mapsto g(f(y_1), x_2)\}$. Of course, one could simplify further and replace $g(f(y_1), x_2)$ by $g(x_2)$, but this will be discussed in more detail later on.

In order to formalize this transformation, we will use some conventions. First, λ denotes an empty sequence (of quantifiers, of variables, etc.). When describing $\mathrm{SL}(\downarrow)$ prefixes we shall use patterns such as

$$\downarrow\forall y_{1|\tau_1} \exists x_{1|\rho_1} \dots \forall y_{k|\tau_k} \exists x_{k|\rho_k} \updownarrow Q;$$

it must be understood that not necessarily all the x_i and y_i are present in the prefix, and that either $\updownarrow Q = \lambda$ or else $\updownarrow \in \{\downarrow, \uparrow\}$ and Q is a (possibly empty) $\mathrm{SL}(\downarrow)$-prefix.

Definition 13. Given $Q\psi$ in prenex normal form (ψ quantifier-free), $T(Q\psi)$ is the SO-formula $\pi(\downarrow Q)[\psi\sigma_{\downarrow Q}]$ with $\sigma_{\downarrow Q} = \tau_\lambda^\lambda(\{\}, \downarrow Q)$, where $\pi(\lambda) = \lambda$, $\tau_\beta^\alpha(\sigma, \lambda) = \sigma$ and

$$\begin{aligned}
\pi(\downarrow\forall y_{1|\tau_1} \exists x_{1|\rho_1} \dots \forall y_{k|\tau_k} \exists x_{k|\rho_k} \updownarrow Q) &= \exists f_{x_1} \dots \exists f_{x_k} \forall y_1 \dots \forall y_k \pi(\updownarrow Q) \\
\pi(\uparrow\exists x_{1|\rho_1} \forall y_{1|\tau_1} \dots \exists x_{k|\rho_k} \forall y_{k|\tau_k} \updownarrow Q) &= \forall g_{y_1} \dots \forall g_{y_k} \exists x_1 \dots \exists x_k \pi(\updownarrow Q) \\
\tau_\beta^\alpha(\sigma, \downarrow\forall y_{1|\tau_1} \exists x_{1|\rho_1} \dots \forall y_{k|\tau_k} \exists x_{k|\rho_k} \updownarrow Q) &= \tau_{\beta'}^{\alpha'}(\sigma \cup \{x_i \mapsto f_{x_i}(\alpha' \setminus \rho_i)\sigma\}, \updownarrow Q) \\
\tau_\beta^\alpha(\sigma, \uparrow\exists x_{1|\rho_1} \forall y_{1|\tau_1} \dots \exists x_{k|\rho_k} \forall y_{k|\tau_k} \updownarrow Q) &= \tau_{\beta'}^{\alpha'}(\sigma \cup \{y_i \mapsto g_{y_i}(\beta' \setminus \tau_i)\sigma\}, \updownarrow Q)
\end{aligned}$$

Here, we assumed $\alpha' := \alpha, y_1 \dots y_k$ and $\beta' := \beta, x_1 \dots x_k$.

[2] As is customary, we use postfix notation for substitution application.

The reader should verify that, modulo variable names, $T(\chi_2) = \chi_2''$. In particular, substitution application in $f_{x_i}(\alpha' \setminus \rho_i)\sigma$ and $g_{y_i}(\beta' \setminus \rho_i)\sigma$ account for the introduction of nested terms like $g(x_2, f(y_1))$ in (13).

Lemma 17. *For every $\varphi \in \mathrm{SL}(\downarrow)$ in prenex normal form, every model $\mathcal{A}$ and every suitable valuation v, $\mathcal{A} \models^+ \varphi[v]$ iff $\mathcal{A} \models_{\mathrm{SO}} T(\varphi)[v]$.*

Proof. First, observe that $\mathcal{A} \models^+ \varphi[v]$ iff $\mathcal{A} \models^+ {\downarrow}\varphi[v]$ (Proposition 1). One then can show that, for every ψ in prenex normal form and every suitable v, $\mathcal{A} \models^+ {\downarrow}\psi[v]$ iff $\mathcal{A} \models_{\mathrm{SO}} T({\downarrow}\psi)[v]$ by induction on the number of turns in $\mathsf{G}(\mathcal{A}, {\downarrow}\psi, \{v\})$ (i.e., in the number of $\downarrow$ and $\uparrow$ occurring in ${\downarrow}\psi$).

The base case is as follows. Suppose φ is of the form

$$\downarrow \forall y_{1|\tau_1} \exists x_{1|\rho_1} \ldots \forall y_{k|\tau_k} \exists x_{k|\rho_k} \psi,$$

where ψ is quantifier-free. In the game $\mathsf{G}(\mathcal{A}, \varphi, \{v\})$, Eloïse has a winning strategy if she can decorate her nodes (i.e. those containing an existential quantifier) such that for any decoration of Abélard's nodes, she wins the game $\mathsf{G}(\mathcal{A}, \psi, \{v\})$. She must decorate the node $\exists x_{i|\rho_i}$ with an $|\mathcal{A}|$-valued function depending on all the variables in the scope of this node in the syntactic tree of φ. Because of Nature's action in the evaluation phase, this function turns out to be equivalent to a one whose value is independent on the values of the variables in ρ_i. Hence Eloïse has a winning strategy if and only if for each i, there is a function f_{x_i} which depends only on the variables in the context of $\exists x_{i|\rho_i}$ except the variables in ρ_i and such that for any choice y_j of Abélard in his node $\forall y_{j|\tau_j}$, the SO formula

$$\varphi' = \exists f_{x_1} \ldots \exists f_{x_k} \forall y_1 \ldots \forall y_k \tilde{\psi}$$

is true in $\mathcal{A}$ under v, where $\tilde{\psi}$ is the result of replacing x_i by $f_{x_i}(\vec{z})$, for the adequate $\vec{z}$, as explained above. One can verify that $T(\varphi) = \varphi'$. Observe that all the existential quantifiers are in front of all the universal quantifiers because Eloïse has to play first (as she owns the initial $\downarrow$ node), and Abélard has to do it in second place. Observe also that it suffices to consider first order universal quantifiers because we are analyzing the case when Eloïse has a winning strategy, and hence Abélard can play 'at random', in the sense that any Eloïse should beat any possible play of Abélard's (see Remark 3).

The analysis for the inductive step is analogous to the basic case, since the game proceeds in turns which are pairwise independent —except from the fact that the valuation is extended at each step. In the same way one can show the dual case of φ starting with $\uparrow \exists x_{1|\rho_1} \forall y_{1|\tau_1} \ldots \exists x_{k|\rho_k} \forall y_{k|\tau_k} \psi$. $\square$

Proof of item 1 of Theorem 16. It follows directly from Definition 13 that for every $\mathrm{SL}(\downarrow)$-formula φ in prenex normal form, $T(\varphi)$ is a SO^w-formula (incidentally, also prenex normal form). This, together with Lemma 17, concludes the proof of the first part of Theorem 16. $\square$

For the proof of item 2 of Theorem 16, we define a translation S that maps SO^w-formulas in prenex normal form with only shallow terms (cf. Proposition 12) to equivalent $\mathrm{SL}(\downarrow)$-formulas. If ψ is quantifier-free, then $S(\psi) = \psi$. Now, suppose we are given a formula of the form:

$$\varphi = \exists g_1 \dots \exists g_n \forall f_1 \dots \forall f_m \psi, \tag{14}$$

where $n > 0$, $m \geq 0$ and $\psi \neq \forall h \psi'$. Every occurrence of g_k in ψ is a shallow term $g_k(\bar{t}_k)$, where, $\bar{t}_k$ denotes a tuple of first-order variables whose dimension is the arity of g_k ($\bar{t}_k = \lambda$ when g_k is a first-order variable). Since φ has weak dependencies, $\bar{t}_k$ can only contain variables free in ψ, including those symbols $g_1, \dots, g_n, f_1, \dots, f_m$ of arity zero (i.e. first-order variables). In particular, by Assumption 9, no variable that is bound in ψ may occur in $\bar{t}_k$.

For $k = 1, \dots, n$, let $\bar{t}_k^1, \dots, \bar{t}_k^{l_k}$ denote all the tuples (of first-order variables) such that $g_k(\bar{t}_k^i)$ occurs in ψ. Let $U = \{u_k^i \mid 1 \leq k \leq n, 1 \leq i \leq l_k\}$ be a set of *fresh variables* and let σ_U be a substitution that replaces the term $g_k(\bar{t}_k^i)$ by the first order variable u_k^i. Moreover, let $Y = \{\bar{y}_k^i \mid 1 \leq k \leq n, 1 \leq i \leq l_k\}$ be a set of *tuples of fresh variables* where the dimension of each $\bar{y}_k^i$ coincides with the arity of g_k. We then define $S(\varphi)$ as:

$$\begin{aligned} S(\varphi) = &\downarrow \forall \bar{y}_1^1 \dots \forall \bar{y}_1^{l_1} \forall \bar{y}_2^1 \dots \forall \bar{y}_2^{l_2} \dots \forall \bar{y}_n^1 \dots \forall \bar{y}_n^{l_n} \\ &\exists u^1_{1|Y \setminus \{\bar{y}_1^1\}} \dots \exists u^{l_1}_{1\;|Y \setminus \{\bar{y}_1^{l_1}\}} \\ &\exists u^1_{2|Y \setminus \{\bar{y}_2^1\}} \dots \exists u^{l_2}_{2\;|Y \setminus \{\bar{y}_2^{l_2}\}} \\ &\dots \\ &\exists u^1_{n|Y \setminus \{\bar{y}_n^1\}} \dots \exists u^{l_n}_{n\;|Y \setminus \{\bar{y}_n^{l_n}\}} [\rho \wedge (\neg \chi \vee S(\forall f_1 \dots \forall f_m [\psi \sigma_U]))] \end{aligned} \tag{15}$$

where $\rho = \bigwedge_{\substack{1 \leq k \leq n, \\ 1 \leq i \leq l_k}} (\bar{y}_k^i \approx \bar{y}_k^j \rightarrow u_k^i \approx u_k^j)$ and $\chi = \bigwedge_{\substack{1 \leq k \leq n, \\ 1 \leq i \leq l_k}} \bar{y}_k^i \approx \bar{t}_k^i$.

Observe that if $\bar{y}_k^i = \langle \bar{y}_k^i(1), \dots \bar{y}_k^i(l_k) \rangle$ and $\bar{y}_k^j = \langle \bar{y}_k^j(1), \dots \bar{y}_k^j(l_k) \rangle$ then $\forall \bar{y}_k^i$ is to be read as $\forall y_k^1(1) \dots \forall y_k^1(l_k)$ while formula $\bar{y}_k^i \approx \bar{y}_k^j$ in ρ is short for $\bigwedge_{1 \leq r \leq l_k} y_k^i(r) \approx y_k^j(r)$. The same applies to $\bar{y}_k^i \approx \bar{t}_k^i$ in χ.

There are two key points in the above definition of $S(\varphi)$. One is that the substitution σ_U eliminates *every* occurrence of g_k in ψ. The other is that in the recursive use of S we use a less complex SO^w-formula (in particular, the $\forall$-prefix may be of length zero). The dual case (i.e., that when φ starts with a $\forall$) is analogous.

Lemma 18. *If x is a first-order variable and φ is a* SO^w*-formula then $S(\exists x \varphi) \equiv_c \downarrow \exists x S(\varphi)$ and $S(\forall x \varphi) \equiv_c \uparrow \forall x S(\varphi)$.*

Proof. We show $S(\exists x \varphi) \equiv_c \downarrow \exists x S(\varphi)$. The case when φ starts with $\forall$ is immediate. Suppose φ is of the form (14), i.e.

$$\varphi = \exists g_1 \dots \exists g_n \forall f_1 \dots \forall f_m \psi,$$

and assume g_1 is of arity 0. Using the nomenclature used above, we have that $l_1 = 1$, since the only tuple $\bar{t}$ such that $g_1(t)$ occurs in ψ is the empty tuple $\bar{t} = ()$; on the other hand, since the dimension of $\bar{y}_1^1$ coincides with the arity of g_1, $\bar{y}_1^1$ gets trivialized to the empty tuple of variables. So (15) becomes

$$\begin{aligned}S(\varphi) = &\downarrow\forall \bar{y}_2^1 \dots \forall \bar{y}_2^{l_2} \dots \forall \bar{y}_n^1 \dots \forall \bar{y}_n^{l_n}\\ &\exists u^1_{1|Y}\\ &\exists u^1_{2|Y\setminus\{\bar{y}_2^1\}} \dots \exists u^{l_2}_{2\ |Y\setminus\{\bar{y}_2^{l_2}\}}\\ &\dots\\ &\exists u^1_{n|Y\setminus\{\bar{y}_n^1\}} \dots \exists u^{l_n}_{n\ |Y\setminus\{\bar{y}_n^{l_n}\}}[\rho \wedge (\neg\chi \vee S(\forall f_1 \dots \forall f_m[\psi\sigma_U]))]\end{aligned} \tag{16}$$

Now the first existential quantifier is independent of all the previous variables quantified universally (i.e. those variables in Y). Then one can swap the block $\forall \bar{y}_2^1 \dots \forall \bar{y}_2^{l_2} \dots \forall \bar{y}_n^1 \dots \forall \bar{y}_n^{l_n}$ and $\exists u^1_{1|Y}$ without changing the meaning. So (16) is equivalent to

$$\begin{aligned}&\downarrow\exists u^1_{1|Y}\\ &\ \forall \bar{y}_2^1 \dots \forall \bar{y}_2^{l_2} \dots \forall \bar{y}_n^1 \dots \forall \bar{y}_n^{l_n}\\ &\ \exists u^1_{2|Y\setminus\{\bar{y}_2^1\}} \dots \exists u^{l_2}_{2\ |Y\setminus\{\bar{y}_2^{l_2}\}}\\ &\ \dots\\ &\ \exists u^1_{n|Y\setminus\{\bar{y}_n^1\}} \dots \exists u^{l_n}_{n\ |Y\setminus\{\bar{y}_n^{l_n}\}}[\rho \wedge (\neg\chi \vee S(\forall f_1 \dots \forall f_m[\psi\sigma_U]))],\end{aligned}$$

which is clearly equivalent in classical contexts to $S(\exists g_2 \dots \exists g_n \forall f_1 \dots \forall f_m \psi)$.

The case $S(\forall x\varphi) \equiv_c \uparrow\forall x S(\varphi)$ is analogous. This concludes the proof of Lemma 18. □

Proof of item 2 of Theorem 16. We prove that $\mathcal{A} \models_{\mathrm{SO}} \varphi[v]$ iff $\mathcal{A} \models^+ S(\varphi)[v]$ by induction on the number of quantifier alternations in φ. The property is trivially true when φ is quantifier-free, so assume instead that φ is of the form (14) –the dual case being analogous. Clearly, φ is equivalent to:

$$\exists g_1 \dots \exists g_n \forall \bar{y}_1^1 \dots \forall \bar{y}_1^{l_1} \forall \bar{y}_2^1 \dots \forall \bar{y}_2^{l_2} \dots \forall \bar{y}_n^1 \dots \forall \bar{y}_n^{l_n}[\neg\chi \vee \forall f_1 \dots \forall f_m[\psi\tau]] \tag{17}$$

where τ is a substitution of $g_k(\bar{t}_k^i)$ by $g_k(\bar{y}_k^i)$. This means that all occurrences of g_k in $\psi\tau$ are shallow. Furthermore, (17) is equivalent to φ' defined as:

$$\exists g_1 \dots \exists g_n \forall \bar{y}_1^1 \dots \forall \bar{y}_1^{l_1} \forall \bar{y}_2^1 \dots \forall \bar{y}_2^{l_2} \dots \forall \bar{y}_n^1 \dots \forall \bar{y}_n^{l_n} \forall f_1 \dots \forall f_m \tilde{\psi} \tag{18}$$

with $\tilde{\psi}$ a SO^w-formula in prenex normal form, equivalent to $\neg\chi \vee (\psi\tau)$ and with the same number of quantifier alternations as $\psi\tau$ (Proposition 12). A simple inspection shows that φ' is a SO^w-formula as well. Call φ'' the result of stripping the $\exists$-prefix from φ'. Then, $\mathcal{A} \models_{\mathrm{SO}} \varphi'[v]$ iff there are functions $G_k : |\mathcal{A}|^{l_k} \to |\mathcal{A}|$ interpreting all the g_k, such that, for $\mathcal{A}' = (\mathcal{A}, G_1, \dots, G_k)$, $\mathcal{A}' \models \varphi''[v]$, iff (inductive hypothesis) $\mathcal{A}' \models^+ S(\varphi'')[v]$, iff (Lemma 18):

$$\mathcal{A}' \models^+ \downarrow\forall \bar{y}_1^1 \dots \downarrow\forall \bar{y}_1^{l_1} \downarrow\forall \bar{y}_2^1 \dots \downarrow\forall \bar{y}_2^{l_2} \dots \downarrow\forall \bar{y}_n^1 \dots \downarrow\forall \bar{y}_n^{l_n} S(\forall f_1 \dots \forall f_m \tilde{\psi})[v] \tag{19}$$

which holds (using again the identity $\downarrow\exists x \downarrow\chi \equiv_c \downarrow\exists x\chi$) iff:

$$\mathcal{A}' \models^+ \downarrow\forall \bar{y}_1^1 \dots \forall \bar{y}_1^{l_1} \forall \bar{y}_2^1 \dots \forall \bar{y}_2^{l_2} \dots \forall \bar{y}_n^1 \dots \forall \bar{y}_n^{l_n} S(\forall f_1 \dots \forall f_m \tilde{\psi})[v] \tag{20}$$

Now, the key observation is that (20) holds iff $\mathcal{A} \models^+ \gamma[v]$, where γ is:

$$\begin{aligned}
&{\downarrow}\forall \bar{y}_1^1 \dots \forall \bar{y}_1^{l_1} \forall \bar{y}_2^1 \dots \forall \bar{y}_2^{l_2} \dots \forall \bar{y}_n^1 \dots \forall \bar{y}_n^{l_n} \\
&\ \exists u_{1|Y\setminus\{\bar{y}_1^1\}}^1 \dots \exists u_{1\ |Y\setminus\{\bar{y}_1^{l_1}\}}^{l_1} \\
&\ \exists u_{2|Y\setminus\{\bar{y}_2^1\}}^1 \dots \exists u_{2\ |Y\setminus\{\bar{y}_2^{l_2}\}}^{l_2} \\
&\ \dots \\
&\ \exists u_{n|Y\setminus\{\bar{y}_n^1\}}^1 \dots \exists u_{n\ |Y\setminus\{\bar{y}_n^{l_n}\}}^{l_n} [\rho \wedge S(\forall f_1 \dots \forall f_m[\tilde{\psi}\tau])\tau']
\end{aligned} \tag{21}$$

and τ' substitutes all the occurrences of $g(\bar{y}_k^i)$ by u_k^i. We prove this claim next, but notice that $S\left(\forall f_1 \dots \forall f_m[\tilde{\psi}\tau]\right)\tau' = S\left(\forall f_1 \dots \forall f_m[\tilde{\psi}\tau\tau']\right)$, and $\tilde{\psi}\tau\tau' \equiv \psi\sigma_U$. Then $S(\forall f_1 \dots \forall f_m[\tilde{\psi}\tau\tau']) \equiv_c S(\forall f_1 \dots \forall f_m[\psi\sigma_U])$, and we conclude that $\gamma \equiv_c S(\varphi)$, which would conclude the proof.

Then, $\mathcal{A} \models^+ \gamma[v]$ iff, according to the block of existential quantifiers $\exists u_k^i$, Eloïse has functions $\tilde{G}_k^i$ to play in the nodes for u_k^i for $k = 1, \dots, n$ and $i = 1, \dots, l_k$ *depending only on* $\bar{y}_k^i$ such that makes her win the rest of the game. In any winning strategy for Eloïse, all the $\tilde{G}_k^i$ for a fixed k must be in fact the same function (otherwise, Abélard can beat hear by picking values that distinguish them and playing the proper conjunct in ρ). So if Eloïse has a winning strategy for $\mathsf{G}(\mathcal{A}, \gamma, \{v\})$, her functions $\tilde{G}_k^i$ played in nodes u_k^i show the existence of the functions G_k^i (take $G_k^i := \tilde{G}_k^i$). Conversely, if there exists the functions G_k^i then these constitute the winning strategy for Eloïse (take $\tilde{G}_k^i := G_k^i$). □

5 The connection with the logic of Henkin quantifiers

We have shown in the preceding sections that when one adds classical negation to SL (or IF, for that matter) in the way suggested by Hodges [6], one lands in a rather weak fragment of SO. But this way of incorporating negation may seem arbitrary, so one may wonder whether this was a sensible definition in the first place. We will argue in this section that this is indeed the case. To see this, we resort to Henkin quantifiers.

As an example of the simplest (non-trivial) Henkin quantifier, consider formula:

$$\begin{pmatrix} \forall x_0 & \exists y_0 \\ \forall x_1 & \exists y_1 \end{pmatrix} \varphi(x_0, x_1, y_0, y_1). \tag{22}$$

Semantics are usually given using a Skolemization; for instance, (22) is equivalent to the Σ_1^1-formula $\exists f \exists g \forall x_0 \forall x_1 \varphi(x_0, x_1, f(x_0), f(x_1))$.

More generally (we follow here the presentation in [17]), a *Henkin prefix* can be defined as a triple $Q = \langle A_Q, E_Q, D_Q \rangle$ where A_Q and E_Q are disjoint sets of variables (*universal* and *existential*, respectively) Q, and $D_Q \subseteq A_Q \times E_Q$ is a *dependency relation.* When $(y, x) \in D_Q$, we say that the existential variable y depends on the universal variable x in Q. Moreover, if D_Q is the union of n complete bipartite graphs, then we can write Q in matritial form using n rows, as in (22). L^* is the extension of FO with Henkin prefixes (i.e., they may occur wherever a first-order quantifier is allowed to occur) while L_1^* is the fragment composed by formulas of the form $Q\psi$ with ψ first-order.

The *Skolemization of* $Q\varphi$, denoted $\mathrm{sk}(Q\varphi)$, is defined as the result of substituting in φ all the free occurrences $y_i \in E_Q$ by $f_{y_i}(\bar{x}_i)$, where $\bar{x}_i$ is a tuple containing every x such that $(y_i, x) \in D_Q$. All the f_{y_i} are assumed fresh. We then define $\models_{L^*}$ by extending $\models_{\mathrm{FO}}$ with the semantic clause:

> $\mathcal{A} \models_{L^*} Q\varphi[v]$ iff $(\mathcal{A}, F_1, \ldots, F_k) \models_{L^*} \forall\bar{x}\,\mathrm{sk}(Q\varphi)[v]$, for some functions $F_1, \ldots, F_k$ on $\mathcal{A}$ interpreting the $f_{y_1}, \ldots, f_{y_k}$ introduced by Skolemization.

The reason we are interested in these logics comes from the following well-known result, a direct corollary of the equivalences of Σ_1^1 with L_1^* [4, 18] and with IF [2].

Theorem 19. IF is equivalent to L_1^*.

The crucial connection is that L^* can be regarded as the result of adding classical negation to L_1^*, in an iterated way, in exactly the same way as SL($\downarrow$) is obtained from SL (for a formal treatment of this notion, cf. [19]). We then have the following:

Theorem 20. *L^* is equivalent to* SO^w *and, therefore, to* SL($\downarrow$) *as well.*

Proof. It is straightforward to encode the semantics of an L^*-formula φ as a SO-formula $T(\varphi)$. The only case that introduces second-order quantification is $T(Q\psi) = \exists f_{y_1} \ldots \exists f_{y_k} \forall x_1 \ldots \forall x_l\, T(\mathrm{sk}(Q\psi))$ and one only needs to observe that if $f_{y_i}(\bar{t})$ occurs in $\mathrm{sk}(Q\psi)$ then $\bar{t}$ is a tuple of variables among $x_1 \ldots x_l$. Hence, $T(\varphi)$ is in SO^w.

The argument for the other inclusion is a straightforward modification of the proof of Theorem 16, item 2; one simply needs to replace γ defined in (21) by:

$$\begin{pmatrix} \forall\bar{y}_1^1 & \exists u_1^1 & \ldots & \exists u_1^{l_1} \\ \forall\bar{y}_2^1 & \exists u_2^1 & \ldots & \exists u_2^{l_2} \\ \vdots & \vdots & \ddots & \vdots \\ \forall\bar{y}_n^1 & \exists u_n^1 & \ldots & \exists u_n^{l_n} \end{pmatrix} [\rho \wedge S(\forall f_1 \ldots \forall f_m[\tilde{\psi}\tau])\tau'] \tag{23}$$

and then argue in an analogous way. □

6 An aside: truth-definitions in the analytic hierarchy

The results of this section appear in [7, 8]. We include them in this aside section for the sake of completeness and readability.

Let $\mathbb{N}$ be the standard model of Peano Arithmetic over the signature $\sigma = \langle 0, 1, +, \times \rangle$. For every $n > 0$, let σ_n denote the extension of σ with *unary* function symbols $f_1 \ldots f_n$. We say that a Σ_n^1-formula is *normalized* if it has the form $\exists f_1 \forall f_2 \exists f_3 \ldots Q f_n \psi$ where ψ is a FO formula over σ_n. Notice that every quantifier is immediately followed by its dual so $Q = \forall$ iff n is even.

Proposition 21. *Over $\mathbb{N}$, every Σ^1_n-formula is effectively equivalent to a normalized Σ^1_n-formula.*

Proof. The is a standard result that follows from the expressibility in $\mathbb{N}$ of a pairing function [20]. □

We assume a computable Gödelization that assigns a number $\ulcorner\alpha\urcorner$ to every second-order term or formula α (over signature σ, assuming any f_i may occur as a second-order variable). Moreover, we assume the usual formula manipulation functions; thus, given $\ulcorner\varphi\urcorner$, we will write, $\ulcorner\neg\varphi\urcorner$ for the Gödel number of the formula that results of negating φ, $\ulcorner\exists x\varphi\urcorner$ for the Gödel number of the formula that results from prepending $\exists x$ to φ. Furthermore, sometimes we will mix logical symbols and natural numbers, as in $\exists xp$, for $x, p \in \mathbb{N}$, to denote the formula $\exists y\varphi$ where $\ulcorner y\urcorner = x$ and $\ulcorner\varphi\urcorner = p$.

For every $n > 0$ we assume the following primitive recursive predicates and functions:

1. $\mathrm{Var}(x)$ holds iff x is the Gödel number of a first-order variable.
2. $\mathrm{Trm}_{\sigma_n}(t)$ holds iff t is the Gödel number of a closed term over σ_n.
3. $\mathrm{Frm}^0_{\sigma_n}(p)$ holds iff p is the Gödel number of a FO-formula over σ_n.
4. $\mathrm{Snt}^0_{\sigma_n}(p)$ holds iff p is the Gödel number of a FO-sentence over σ_n.
5. $\mathrm{Snt}^1_{\sigma}(p)$ holds iff p is the Gödel number of a SO-sentence over σ.
6. $\mathrm{In}_{\Sigma^1_\mathrm{n}}(p)$ holds iff $\ulcorner\exists fp\urcorner$ has at most $n-1$ SO-quantifier alternations.
7. $\mathrm{sub}_\sigma(\ulcorner\varphi\urcorner, \ulcorner x\urcorner, t) = \ulcorner\varphi[x/\underline{t}]\urcorner$ if $\mathrm{Frm}^0_\tau(\ulcorner\varphi\urcorner)$ holds ($\underline{t}$ is the numeral of t).
8. $\mathrm{matrix}(\ulcorner\varphi\urcorner) = \ulcorner\psi\urcorner$ if $\mathrm{In}_{\Sigma^1_\mathrm{n}}(\ulcorner\varphi\urcorner)$ holds and ψ is the matrix of φ.
9. $\mathrm{norm}_{\Sigma^1_n}(\ulcorner\varphi\urcorner) = \ulcorner\psi\urcorner$, ψ is a normalized Σ^1_n-formula equivalent to φ.

Since these are primitive recursive, they can be expressed in the FO-language of σ. Fix two second-order variables v and X and let $T_n(v, X)$ be the conjunction of the formulas in Table 1. All but S1 are the standard way of describing, in FO, a truth-predicate X for FO-sentences over σ_n and a valuation function v for closed FO-terms over σ_n (see, e.g. [21]). S1, on the other hand, looks rather unusual and is the only axiom referring to Σ^1_n-sentences. Finally, define the SO-formulas over σ_n:

$$\theta^\exists_n(x) := \exists X\exists v(T_n(v, X) \wedge X(x))$$
$$\theta^\forall_n(x) := \forall X\forall v(T_n(v, X) \to X(x))$$

Lemma 22. *Let φ be a Σ^1_n-sentence over σ, let ψ be its Σ^1_n normalized form and let ψ_0 be the matrix of ψ. Moreover, let θ_n be any of $\theta^\exists_n$ or $\theta^\forall_n$. For any $F_1, \ldots, F_n$ interpreting $f_1, \ldots, f_n$, the following are equivalent:*

T1. $v(\ulcorner 0 \urcorner) \approx 0 \wedge v(\ulcorner 1 \urcorner) \approx 1$

T2. $\forall s \forall t[\mathrm{Trm}_{\sigma_n}(s) \wedge \mathrm{Trm}_{\sigma_n}(t) \rightarrow v(\ulcorner s+t \urcorner) \approx v(s)+v(t)]$

T3. $\forall s \forall t[\mathrm{Trm}_{\sigma_n}(s) \wedge \mathrm{Trm}_{\sigma_n}(t) \rightarrow v(\ulcorner s \times t \urcorner) \approx v(s) \times v(t)]$

T4. $\forall s \forall t[\mathrm{Trm}_{\sigma_n}(s) \wedge \mathrm{Trm}_{\sigma_n}(t) \rightarrow \bigwedge_{i=1}^{n} v(\ulcorner f_i(t) \urcorner) \approx f_i(v(t))]$

E1. $\forall s \forall t[\mathrm{Trm}_{\sigma_n}(s) \wedge \mathrm{Trm}_{\sigma_n}(t) \rightarrow (X(\ulcorner (s \approx t) \urcorner) \leftrightarrow v(s) \approx v(t))]$

E2. $\forall s \forall t[\mathrm{Trm}_{\sigma_n}(s) \wedge \mathrm{Trm}_{\sigma_n}(t) \rightarrow (X(\ulcorner (s \not\approx t) \urcorner) \leftrightarrow v(s) \not\approx v(t))]$

B1. $\forall p \forall q[\mathrm{Snt}^0_{\sigma_n}(p) \wedge \mathrm{Snt}^0_{\sigma_n}(q) \rightarrow (X(\ulcorner p \wedge q \urcorner) \leftrightarrow X(p) \wedge X(q))]$

B2. $\forall p \forall q[\mathrm{Snt}^0_{\sigma_n}(p) \wedge \mathrm{Snt}^0_{\sigma_n}(q) \rightarrow (X(\ulcorner p \vee q \urcorner) \leftrightarrow X(p) \vee X(q))]$

Q1. $\forall p \forall x[\mathrm{Frm}^0_{\sigma_n}(p) \wedge \mathrm{Var}(x) \wedge \mathrm{Snt}^0_{\sigma_n}(\ulcorner \forall x p \urcorner) \rightarrow (X(\ulcorner \forall x p \urcorner) \leftrightarrow \forall i X(\mathrm{sub}_{\sigma}(p,x,i)))]$

Q2. $\forall p \forall x[\mathrm{Frm}^0_{\sigma_n}(p) \wedge \mathrm{Var}(x) \wedge \mathrm{Snt}^0_{\sigma_n}(\ulcorner \exists x p \urcorner) \rightarrow (X(\ulcorner \exists x p \urcorner) \leftrightarrow \exists i X(\mathrm{sub}_{\sigma}(p,x,i)))]$

S1. $\forall p[\mathrm{Snt}^1_{\sigma}(p) \wedge \mathrm{In}_{\Sigma^1_{\mathrm{n}}}(p) \rightarrow (X(p) \leftrightarrow X(\mathrm{norm}_{\Sigma^1_n}(p)) \leftrightarrow X(\mathrm{matrix}(\mathrm{norm}_{\Sigma^1_n}(p))))]$

Table 1: $T_n(v, X)$ is the conjunction of these formulas. X is used as a unary predicate but can be assumed to be a unary function with image in $\{0, 1\}$.

1. $(\mathbb{N}, F_1, \ldots, F_n) \models_{\mathrm{FO}} \psi_0$

2. $(\mathbb{N}, F_1, \ldots, F_n) \models_{\mathrm{SO}} \theta_n(\ulcorner \psi_0 \urcorner)$

3. $(\mathbb{N}, F_1, \ldots, F_n) \models_{\mathrm{SO}} \theta_n(\ulcorner \psi \urcorner)$

4. $(\mathbb{N}, F_1, \ldots, F_n) \models_{\mathrm{SO}} \theta_n(\ulcorner \varphi \urcorner)$

Proof. We show only the case for $\theta_n = \theta^{\exists}_n$. We first show that if either 2, 3 or 4 hold, then 1 holds as well. Let $\chi \in \{\varphi, \psi, \psi_0\}$ and assume that $(\mathbb{N}, F_1, \ldots, F_n) \models_{\mathrm{SO}} \theta^{\exists}_n(\ulcorner \chi \urcorner)$. That means that for some $\tilde{X}$ and $\tilde{v}$, $(\mathbb{N}, F_1, \ldots, F_n, \tilde{X}, \tilde{v}) \models_{\mathrm{FO}} T_n(v, X) \wedge X(\ulcorner \chi \urcorner)$. By S1, we may conclude that $(\mathbb{N}, F_1, \ldots, F_n, \tilde{X}, \tilde{v}) \models_{\mathrm{FO}} T_n(v, X) \wedge X(\ulcorner \psi_0 \urcorner)$; moreover all the other formulas make $\tilde{v}$ and $\tilde{X}$ uniquely determined on the Gödel number of closed terms and sentences over σ_n, respectively. Hence, we conclude that $(\mathbb{N}, F_1, \ldots, F_n) \models_{\mathrm{FO}} \psi_0$.

Now we show that if 1 holds, then 2, 3 and 4 hold too. Assume then that $(\mathbb{N}, F_1, \ldots, F_n) \models_{\mathrm{FO}} \psi_0$. We know that $F_1, \ldots, F_n$ induce a unique valuation γ on closed terms and a unique set Ψ_0 of FO-formulas over σ_n that are true in $(\mathbb{N}, F_1, \ldots, F_n)$, from which $\psi_0 \in \Psi_0$ by assumption. Moreover, Ψ_0 induces the unique set Ψ of normalized Σ^1_n-formulas such that $\chi \in \Psi$ iff the matrix of χ is in Ψ_0, so $\psi \in \Psi$. Finally, Ψ induces the unique set of Σ^1_n-formulas Φ such that $\chi \in \Phi$ iff its normalized Σ^1_n-form is in Ψ, from which $\varphi \in \Phi$. Let $\tilde{v}$ be any function such that, for any closed term t over σ_n, $\tilde{v}(\ulcorner t \urcorner) = \gamma(t)$ and let $\tilde{X} = \{\ulcorner \chi \urcorner \mid \chi \in \Psi_0 \cup \Psi \cup \Phi\}$. By construction, we have that $(\mathbb{N}, F_1, \ldots, F_n, \tilde{X}, \tilde{v}) \models T_n(v, X) \wedge X(\ulcorner \psi_0 \urcorner) \wedge X(\ulcorner \psi \urcorner) \wedge X(\ulcorner \varphi \urcorner)$, so $(\mathbb{N}, F_1, \ldots, F_n) \models \theta^{\exists}_n(\ulcorner \chi \urcorner)$ for $\chi \in \{\psi_0, \psi, \varphi\}$. □

The main result of this section is that the truth-predicate for Σ^1_n-sentences over $\mathbb{N}$ is a Σ^1_n-set in the analytic hierarchy. This is formally stated as follows:

Theorem 23. *For all $n > 0$ there is a Σ^1_n-formula $\tau_n(x)$ over σ such that for every Σ^1_n-sentence φ over σ, $\mathbb{N} \models_{\mathrm{SO}} \varphi$ iff $\mathbb{N} \models_{\mathrm{SO}} \tau_n(\lceil\varphi\rceil)$.*

Proof. We discuss the case for n odd, the even case being analogous. Define, then $\tau_n(x) := \exists f_1 \forall f_2 \ldots \exists f_n \theta^{\exists}_n(x)$, which is clearly Σ^1_n since $\theta^{\exists}_n(x)$ is Σ^1_n. For the "only if" case, assume $\mathbb{N} \models_{\mathrm{SO}} \varphi$, which implies $\mathbb{N} \models_{\mathrm{SO}} \exists f_1 \forall f_2 \ldots \exists f_n \psi_0$ where ψ_0 is the matrix of a normalized Σ^1_n-sentence equivalent to φ. This means there is a strategy for Eloïse in the standard game-semantics for SO that allows her to reach to a position such that $(\mathbb{N}, F_1, \ldots, F_n) \models_{\mathrm{FO}} \psi_0$ regardless what Abélard plays. By Lemma 22, this same strategy is also winning when playing over formula $\exists f_1 \forall f_2 \ldots \exists f_n \theta^{\exists}_n(\lceil\varphi\rceil)$, which means that $\mathbb{N} \models_{\mathrm{SO}} \tau_n(\lceil\varphi\rceil)$. The converse case is analogous. □

Now, observe that if $\tau_n(x)$ is the Σ^1_n-truth-predicate for Σ^1_n-sentences, then $\tau'_n(\lceil\varphi\rceil) := \neg\tau_n(\lceil\neg\varphi\rceil)$ is a Π^1_n-truth-predicate for Π^1_n-sentences. Therefore, we get the dual result:

Corollary 24. *For all $n > 0$ there is a Π^1_n-formula $\tau'_n(x)$ over σ such that for every Π^1_n-sentence φ over σ, $\mathbb{N} \models_{\mathrm{SO}} \varphi$ iff $\mathbb{N} \models_{\mathrm{SO}} \tau'_n(\lceil\varphi\rceil)$.*

We say that a logic $\mathcal{L}$ is *arithmetically reducible* to Σ^1_n over σ if there is a function $g_{\Sigma^1_n}$, expressible in FO over σ, such that for any $\mathcal{L}$-formula φ, $g_{\Sigma^1_n}(\varphi)$ is a Σ^1_n-formula and $\mathbb{N} \models_{\mathcal{L}} \varphi$ iff $\mathbb{N} \models_{\mathrm{SO}} g_{\Sigma^1_n}(\varphi)$. The notion of arithmetical reduction to Π^1_n is analogous. It is straightforward to see that if $\tau_n(x)$ is the truth-predicate for Σ^1_n, then $\tau_{\mathcal{L}}(x) = \tau_n(g_{\Sigma^1_n}(x))$ is a Σ^1_n-truth-predicate for $\mathcal{L}$. Again, a similar result holds for the Π^1_n case. From this we get the following result, which allows one to easily prove separation results for a logic with respect to Δ^1_n (i.e., to the set of formulas that are logically equivalent both to a Σ^1_n- and to a Π^1_n-formula).

Theorem 25. *For $n > 1$, if $\mathcal{L}$ is closed under Boolean operations and arithmetically reducible to both Σ^1_n and Π^1_n then there is a Δ^1_n-formula that is not equivalent to any formula in $\mathcal{L}$.*

Proof. Let $\tau_{\mathcal{L}} \subseteq \mathbb{N}$ be such that $\lceil\varphi\rceil \in \tau_{\mathcal{L}}$ iff $\mathbb{N} \models_{\mathcal{L}} \varphi$. By the remark above, $\tau_{\mathcal{L}}$ is a Δ^1_n-set of natural numbers, so let $\tau^{\Sigma}_{\mathcal{L}}(x)$ and $\tau^{\Pi}_{\mathcal{L}}(x)$ be the Σ^1_n- and Π^1_n-predicates defining $\tau_{\mathcal{L}}$. Since $\mathbb{N}$ may be described up-to-isomorphism by a Π^1_1-sentence ψ_{PA} and Σ^1_n and Π^1_n are closed (for $n > 1$) by conjunctions with Π^1_1-formulas we get that $\tau^{\Sigma}_{\mathcal{L}}(x) \wedge \psi_{\mathrm{PA}}$ is a Δ^1_n-formula and by Tarski's Undefinability Theorem, it is not expressible in $\mathcal{L}$. □

We conjecture that this result holds for the case $n = 1$.

7 Discussion

The motivation for the research reported in this paper was an interest in understanding what are the properties of IF with classical negation, as defined by Hodges [6]. Taking as starting point the equivalent, game-theoretical semantics introduced in [9], we first found a characterization of SL(↓) in terms of a syntactic fragment of SO and proved that this fragment is indeed quite weak: it is strictly contained in Δ^1_2. Moreover, this characterization allowed us to precisely locate SL(↓) in the logic spectrum: it corresponds to the well-studied logic of Henkin quantifiers, L^*.

The equivalence with L^* is a pleasant result, in that it gives a concrete and definite answer to our motivating question. It also means that some of the results we obtained in the process can, alternatively, be concluded from known properties of L^*. It is therefore interesting to make a comparison of both approaches.

In retrospect, one finds that Enderton [4] already saw the connection between L^* and the fragment of SO with weak dependencies (cf. Section 5). He goes as far as "cheating" (sic) by saying [4, p. 394]:

> The class of finite partially-ordered (f.p.o) formulas then is defined by adding one additional clause to the definition of elementary formula (with equality): If $\varphi(x_1, \dots, x_m, y_1, \dots, y_n)$ is a f.p.o formula, then so is the formula:
>
> $$\exists F_1 \dots \exists F_n \forall x_1 \dots \forall x_n \varphi(x_1, \dots, x_m, F_1(\vec{x}_1), \dots, F_n(\vec{x}_n))$$
>
> where $\vec{x}_i$ is a sublist of $x_1, \dots x_n$.

He then shows [4, Theorem 2] that any L^* formula can be effectively reduced to a Σ^1_2 and to a Π^1_2 formula (which proves L^* to be, at most, as expressive as Δ^1_2) and the sketched proof corresponds, essentially, to the proof of our Proposition 14. Mostowski, on discussing Enderton's result, says [5, p. 23]:

> This would suggest that the logic of branched quantifiers is equivalent to Δ^1_2-second order formulae. However it is not so.

He then proves the separation employing an ad-hoc truth-predicate construction sketched in [22].

On the contrary, we derive the strict separation of Δ^1_2 and L^* (or SO^w or SL(↓), for that matter) as a trivial corollary of Proposition 14, by way of the results discussed in Section 6 for the analytic hierarchy.

It has been shown in previous work how to extend the expressive power of L^* to cover the whole of SO. The idea, roughly, is to allow quantifiers in which existential variables may depend on universal variables and the other way round as well (see, e.g. [23] for more details).

Another approach, completely different from Hodges', for adding classical negation to IF is considering Team Logic [24]. On the one hand, Team Logic is an extension of Dependence Friendly Logic obtained by adding classical negation. On the other, Team Logic can be seen as the closure of IF under classical negation. In this setting, adding classical negation adds much more expressivity, as the expressive power of Team Logic is equivalent to SO.

One natural question is, then, if there is a counterpart extension for SL(↓) that lands it in full SO. In addition, it would be interesting to find out if SO^w coincides or not with the Boolean closure of Σ^1_1.

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