

Dynamic Modal Logics

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Quote of the Day: “It was all so different before everything changed.”

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 - ▶ Reduction axioms and axiomatization.

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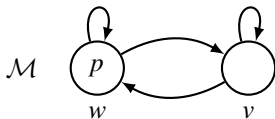
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- ▶ We then move to **more expressive** dynamic logics: Relation Changing Logics and Memory Logics.
 - ▶ Again we will visit some expressivity and complexity results.

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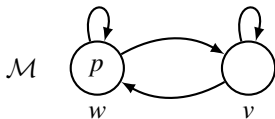
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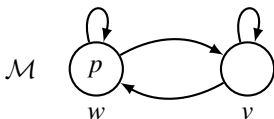
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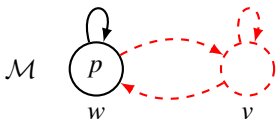
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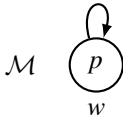
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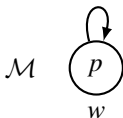
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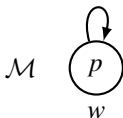
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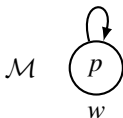
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Can we capture what just happened with a modality?

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Plaza, J., 1989,

Logics of Public Communications,

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Gerbrandy, J. and W. Groeneveld: 1997,
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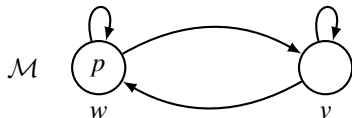
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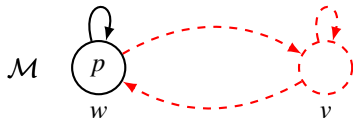
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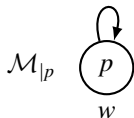
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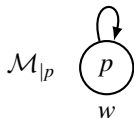
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$[!\psi]$ is a well behaved modality. E.g.

- ▶ $[!\psi]$ -K $\models [!\psi](\varphi \rightarrow \chi) \rightarrow ([!\psi]\varphi \rightarrow [!\psi]\chi)$.
- ▶ $[!\psi]$ -Nec $\models \varphi \text{ then } \models [!\psi]\varphi$.

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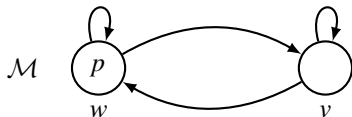
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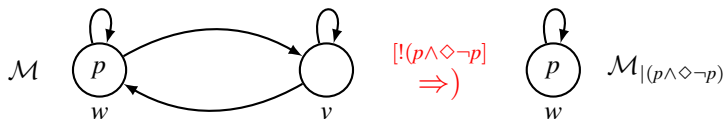
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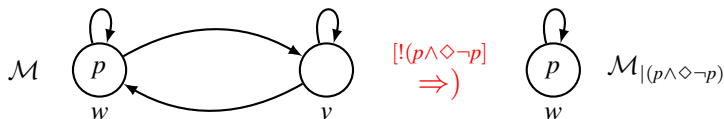
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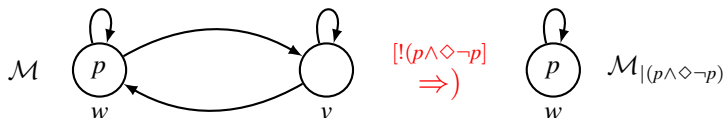
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- ▶ **Moore's Paradox**: after truthfully announcing a fact, it becomes false.

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- ▶ That is, we cannot replace a propositional symbol with arbitrary formulas and preserve validity.
- ▶ Thus, uniform substitution fails.

How do we axiomatize a logic without uniform substitution?

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Definition (**BML**-bisimulation)

Two models $\mathcal{M} = \langle W, R, V \rangle$ and $\mathcal{M}' = \langle W', R', V' \rangle$ are **bisimilar** if there is a nonempty relation $Z \subseteq W \times W'$ s.t. wZw' then

- [atom] w and w' agree on all propositional symbols.
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Theorem (Invariance)

*If \mathcal{M}, w and \mathcal{M}', w' are bisimilar (notation: $\mathcal{M}, w \Leftrightarrow \mathcal{M}', w'$), then every **BML**-formula true at \mathcal{M}, w is true at \mathcal{M}', w' .*

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More precisely:

Theorem

For all $\varphi \in \mathbf{PAL}$ there is $\varphi' \in \mathbf{BML}$ such that for all models \mathcal{M}, w , $\mathcal{M}, w \models \varphi \leftrightarrow \varphi'$.

Reduction Axioms for PAL

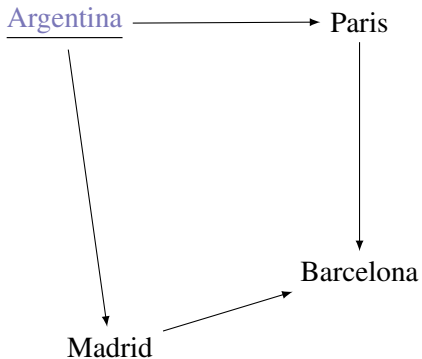
Using **reduction axioms**, we can translate formulas with public announcements into the basic modal language:

1. $[\!|\psi|]p \leftrightarrow (\psi \rightarrow p), p \in \text{PROP}$
2. $[\!|\psi|]\neg\varphi \leftrightarrow (\psi \rightarrow \neg[\!|\psi|]\varphi)$
3. $[\!|\psi|](\varphi \wedge \chi) \leftrightarrow ([\!|\psi|]\varphi \wedge [\!|\psi|]\chi)$
4. $[\!|\psi|]\Box\varphi \leftrightarrow (\psi \rightarrow \Box(\psi \rightarrow [\!|\psi|]\varphi))$

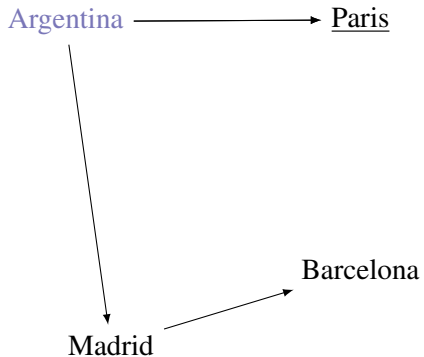
Theorem

*Axioms 1-4 above are sound, and they can be used to obtain, given φ in **PAL**, an equivalent formula in **BML**.*

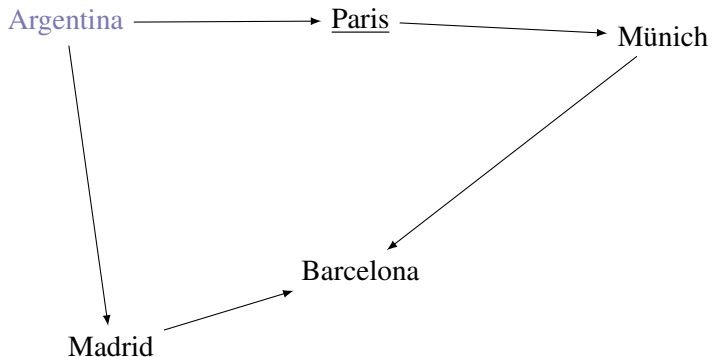
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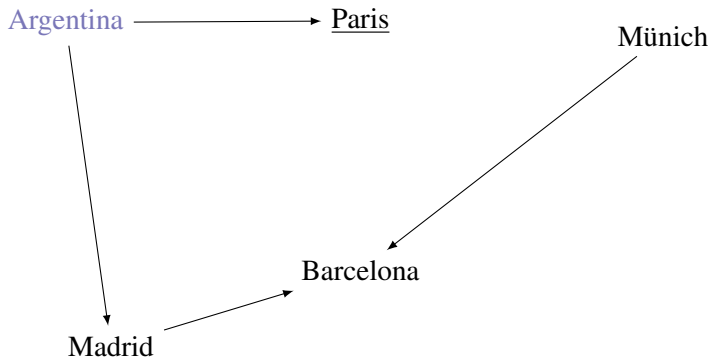
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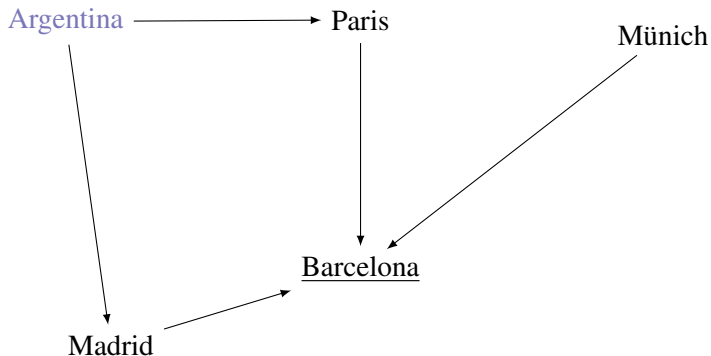
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A New Dynamic Modal Logic

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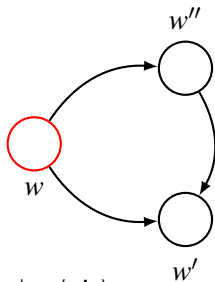


van Benthem, J., 2005,

An Essay on Sabotage and Obstruction,

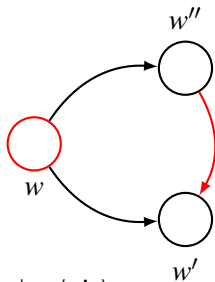
In: D. Hutter and W. Stephan (ed.), *Mechanizing Mathematical Reasoning*, pp. 268–276.

An Example



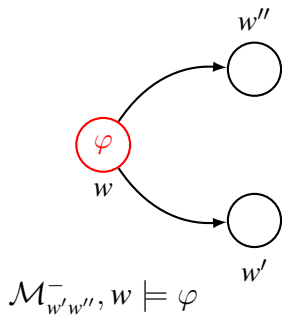
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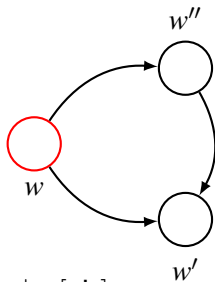


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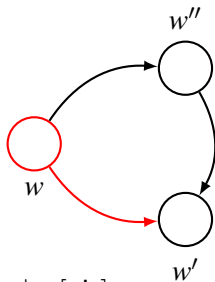


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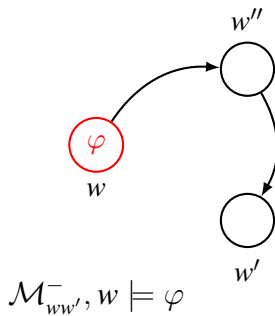
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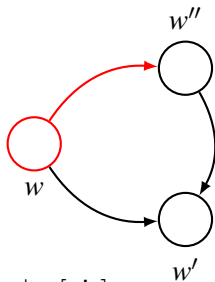


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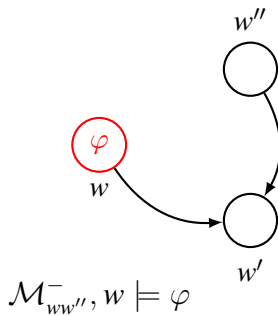


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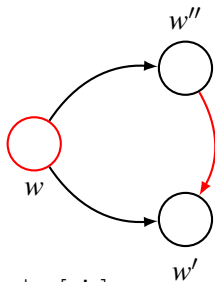


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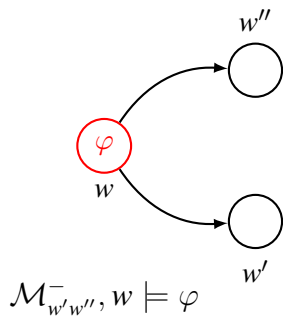


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Other Operators

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For this talk, we will continue working with $\langle \mathbf{sb} \rangle$.



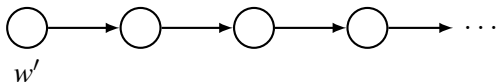
Fervari, R., 2014,

Relation-Changing Modal Logics,

Phd Thesis, Universidad Nacional de Córdoba, Argentina.

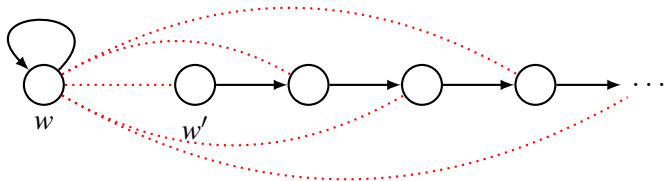
The Expressive Power of $\langle \mathbf{sb} \rangle$: Tree Models

Let's consider the following models:



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The models are bisimilar for **BML**.

Observation: the model on the right is a **tree**.

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Theorem:

BML has the **tree model property**: for every formula φ of **BML**, if φ is satisfiable, then φ is also satisfiable at the root of a tree.

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Corollary.

BML is **strictly less expressive** than **BML**($\langle\langle\text{sb}\rangle\rangle$).

There are no reduction axioms from **BML**($\langle\langle\text{sb}\rangle\rangle$) into **BML**.

Complexity for the MC and SAT Problems

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Theorem (Model Checking)

Given a finite, arbitrary model \mathcal{M} , w and $\varphi \in \mathbf{BML}(\langle \mathbf{sb} \rangle)$, deciding whether $\mathcal{M}, w \models \varphi$ is *PSpace-complete*. (Same as FO, it is in P for **BML**).

Theorem (Satisfiability Checking)

Given $\varphi \in \mathbf{BML}(\langle \mathbf{sb} \rangle)$, deciding whether it has a model (in the class of all models) is *undecidable*. (Same as FO, it is PSpace-complete for **BML**).



C. Areces, R. Fervari, and G. Hoffmann, 2012,
Moving Arrows and Four Model Checking Results,
In: L. Ong and R. de Queiroz, editors, Proc. of WoLLIC 2012, pp. 142–153.



C. Areces, R. Fervari, G. Hoffmann, and M. Martel, 2018,
Satisfiability for Relation-changing Logics,
Journal of Logic and Computation, 28(7):1443–1470.

Complexity for the MC and SAT Problems

- ▶ For MC the upper bound is obtained by **translation into FO**, the lower bound by encoding of **QBF satisfiability**.
- ▶ For SAT we encode the satisfiability problem of a **memory logic**, known to be undecidable.

These results hold for most relation-changing logics we study (e.g., local and global versions, adding, deleting or reversing arrows).

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We know some decidable cases for **BML**($\langle\langle\text{sb}\rangle\rangle$), e.g.

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Open: Exact complexity bounds?

Reasoning with **BML**($\langle \text{sb} \rangle$)

It is easy to see that the set of validities in **BML**($\langle \text{sb} \rangle$) is, again, **not closed under uniform substitution**:

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- ▶ No complete axiomatization of $\mathbf{BML}(\langle \mathbf{sb} \rangle)$ has been given.
- ▶ No sequent or natural deduction system.
- ▶ We do have sound and complete tableaux.

$$\frac{(n, S) : \langle \mathbf{sb} \rangle \varphi}{\begin{array}{c} Rpq \\ pq \notin S \\ (n, S \cup pq) : \varphi \end{array}} \quad ((\mathbf{sb})) p, q \text{ new} \qquad \frac{(n, S) : [\mathbf{sb}] \varphi}{\begin{array}{c} Rpq \\ pq \notin S \\ (n, S \cup pq) : \varphi \end{array}} \quad ([\mathbf{sb}])$$



C. Areces, R. Fervari, and G. Hoffmann, 2013,
Tableaux for Relation-Changing Modal Logics,
In: Proc. of FRODOS 2013.

Memory Logics

Take a relational model $\mathcal{M} = \langle W, R, V \rangle$ and extend it with a **storage** or **memory** $S \subset W$.

Define the following modal operators

$$\begin{aligned} \langle W, R, V, S \rangle, w \models \textcircled{r}\varphi & \text{ iff } \langle W, R, V, S \cup \{w\} \rangle, w \models \varphi \\ \langle W, R, V, S \rangle, w \models \textcircled{k} & \text{ iff } w \in S. \end{aligned}$$

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C. Areces, S. Figueira, and S. Mera.

Completeness results for memory logics,

Annals of Pure and Applied Logic, 163(7):961–972, 2012.



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The Expressive Power of Memory Logics. *Review of Symbolic Logic*, 4(2):290–318, Cambridge University Press, 2011.