Tableaux Methods for Modal and Hybrid Logics

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Before we start...

On Debian/Ubuntu Linux:

```
sudo apt install cabal-install
sudo apt install graphviz
cabal update
cabal install hgen
cabal install HTab
```

HTab's package page:

Plan

1. Semantic-Based Automated Reasoning
2. Tableaux for Propositional Logic
3. Tableaux for Modal Logic
4. Tableaux for Hybrid Logic

Semantic-Based Automated Reasoning

- General goal: mechanize logical reasoning, efficiently.
- Semantic-Based: work with interpretations (models), as opposed to: working with axioms and inference rules.
- Typical reasoning tasks:
  - Prove consequence ($\models \varphi \to \psi$)
  - Prove validity ($\models \varphi$)
  - Prove satisfiability ($\mathcal{M} \models \varphi$ for some $\mathcal{M}$)
    - Model building: actually produce the model $\mathcal{M}$ that satisfies φ
- Main approaches:
  - Resolution
  - Tableaux (also called: semantic tableaux, analytic tableaux)

Semantics vs Proof Systems

	Semantics	**Proof Systems**
Meaning	Truth in models	Rules of derivation
Example	p true in some world	From A and $A \to B$, infer B
Goal	Define when φ is true	Derive φ syntactically

What is a Tableau?

- Systematic **search for a model** for a given formula φ.
- Break down formulas step by step into subformulas, according to its main connector.
- Search space has a tree shape, each branch represents a possible interpretation.
- If all branches **close** (contradiction) $\Rightarrow$ formula valid.
- If one **saturated** branch is **open** $\Rightarrow$ model found.

Signed rules

$$\frac{T\,A \wedge B}{\begin{array}{l} T\,A \\ T\,B \end{array}} \qquad \frac{F\,A \wedge B}{F\,A \;\big|\; F\,B} \qquad \frac{T\,A \vee B}{T\,A \;\big|\; T\,B} \qquad \frac{F\,A \vee B}{\begin{array}{l} F\,A \\ F\,B \end{array}}$$

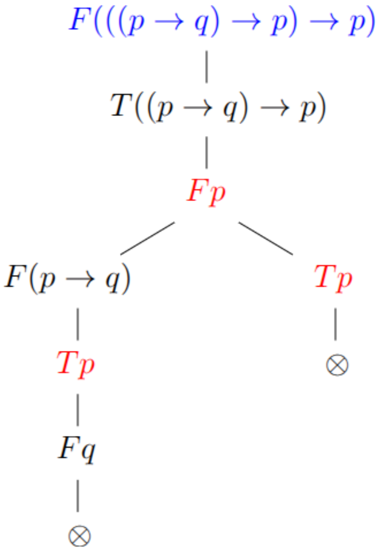
$$\frac{T\,A \rightarrow B}{F\,A \;\big|\; T\,B} \qquad \frac{F\,A \rightarrow B}{\begin{array}{l} T\,A \\ F\,B \end{array}} \qquad \frac{T\,\neg A}{F\,A} \qquad \frac{F\,\neg A}{T\,A}$$

Unsigned rules

$$\frac{A \wedge B}{\begin{array}{l} A \\ B \end{array}} \qquad \frac{\neg(A \wedge B)}{\neg A \;\big|\; \neg B} \qquad \frac{A \vee B}{A \;\big|\; B} \qquad \frac{\neg(A \vee B)}{\begin{array}{l} \neg A \\ \neg B \end{array}}$$

$$\frac{A \rightarrow B}{\neg A \;\big|\; B} \qquad \frac{\neg(A \rightarrow B)}{\begin{array}{l} A \\ \neg B \end{array}} \qquad \frac{\neg\neg A}{A}$$

Example (validity)



Observations

- Truth tables: exhaustive enumeration
- Tableaux: structured semantic search
- Tableaux scale better than truth-table reasoning
- They are **goal-directed** and **systematic**

Kripke model (W, R, V):

- W: set of possible worlds
- $R \subseteq W \times W$: accessibility relation
- $V : PROP \mapsto 2^w$

$$M, w \models \Box\varphi \iff \text{for all } v, (wRv \Rightarrow v \models \varphi)$$
$$M, w \models \Diamond\varphi \iff \text{exists } v : wRv \text{ and } v \models \varphi$$

Idea

Tableaux for modal logic need to handle possible worlds: extend the tableau with **labels** (or prefixes).

Tableaux Rules (Modal Logic)

$$\frac{\sigma{:}(\varphi \wedge \psi)}{\begin{array}{c}\sigma{:}\varphi \\ \sigma{:}\psi\end{array}}\ (\wedge) \qquad \frac{\sigma{:}(\varphi \vee \psi)}{\sigma{:}\varphi \mid \sigma{:}\psi}\ (\vee) \qquad \frac{\sigma{:}\Diamond\varphi}{\begin{array}{c}\sigma{:}\Diamond\tau \\ \tau{:}\varphi\end{array}}\ (\Diamond)^1 \qquad \frac{\begin{array}{c}\sigma{:}\Box\varphi \\ \sigma{:}\Diamond\tau\end{array}}{\tau{:}\varphi}\ (\Box)$$

[1] Prefix τ is new on the branch.

Prefixed formulas: $\sigma{:}\varphi$, $\sigma \in \mathsf{PREF}$
A branch is **closed** if it contains both $\sigma{:}p$ and $\sigma{:}\neg p$.

Examples...

The basic Hybrid Logic $\mathcal{H}(@)$

Recipe

basic modal logic			
+	*nominals*	$\rightarrow$	similar to propositional symbols but with a restricted semantic
+	@	$\rightarrow$	satisfaction operator
	$\mathcal{H}(@)$	$\rightarrow$	basic hybrid logic

- A nominal points to a unique element of the model
- $@_i\varphi$ is true iff φ is true at the element pointed by i

The basic Hybrid Logic $\mathcal{H}(@)$

Syntax

$$\mathsf{FORM} := p \mid i \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \Diamond\varphi \mid @_i\varphi,$$

where $p \in \mathsf{PROP}$, $i \in \mathsf{NOM}$, $\varphi, \varphi_1, \varphi_2 \in \mathsf{FORM}$.

Abbreviations:

- $\Box\varphi = \neg\Diamond\neg\varphi$
- $\varphi \vee \psi = (\neg\varphi) \wedge (\neg\psi)$

The basic Hybrid Logic $\mathcal{H}(@)$

Semantics

Let $\mathcal{M} = \langle W, R, V \rangle$ a model and $w \in W$:

$$
\begin{array}{rcl}
\mathcal{M}, w \models p & \text{iff} & w \in V(p) \text{ for } p \in \mathsf{PROP} \\
\mathcal{M}, w \models \neg\varphi & \text{iff} & \mathcal{M}, w \not\models \varphi \\
\mathcal{M}, w \models \varphi \wedge \psi & \text{iff} & \mathcal{M}, w \models \varphi \text{ and } \mathcal{M}, w \models \psi \\
\mathcal{M}, w \models \Diamond\varphi & \text{iff} & \exists w' \in W \ . \ (w, w') \in R \text{ and } \mathcal{M}, w' \models \varphi \\
 & & \\
\mathcal{M}, w \models i & \text{iff} & V(i) = \{w\} \\
\mathcal{M}, w \models @_i\varphi & \text{iff} & \mathcal{M}, v \models \varphi \text{ , where } V(i) = \{v\}
\end{array}
$$

If $i \in \mathsf{NOM}$, $V(i)$ is a singleton.

Tableaux Rules (part 2)

$$\frac{\sigma{:}\neg a}{\tau{:}a}\,(\neg)^1 \qquad \frac{\sigma{:}@_a\varphi}{\tau{:}a,\, \tau{:}\varphi}\,(@)^1$$

$$\frac{\sigma{:}\varphi,\, \sigma{:}a,\, \tau{:}a}{\tau{:}\varphi}\,(Id) \qquad \frac{\sigma{:}b,\, \sigma{:}a,\, \tau{:}a}{\tau{:}b}\,(nom)$$

[1] Prefix τ is new in the branch.

Branch is closed when $\sigma{:}n$ and $\sigma{:}\neg n$ are in the branch, for $n \in \mathsf{PROP} \cup \mathsf{NOM}$.

Examples...

Model Building

Given an open, saturated branch Θ with root $\sigma_0\varphi_0$, define the model:

$$\mathcal{M}^\Theta = (W^\Theta, R^\Theta, V^\Theta), \quad \text{where}$$

$$\begin{aligned}
W^\Theta &= \{\sigma \mid \sigma \text{ is a prefix in } \Theta\} \\
R^\Theta &= \{(\sigma, s_\Theta\sigma_1) \mid \sigma\Diamond(\sigma_1) \in \Theta\}, \\
V^\Theta(p) &= \{\sigma \mid \sigma{:}p \in \Theta\}, \\
V^\Theta(a) &= \begin{cases} \{\sigma_0\} & \text{if there is no } \sigma \text{ for which } \sigma{:}a \in \Theta, \\ \{s_\Theta(\sigma)\} & \text{if } \sigma{:}a \in \Theta. \end{cases}
\end{aligned}$$

Theorem (Completeness of the hybrid tableaux calculus)

Let Θ be an open, saturated branch in the hybrid tableaux calculus, where $\sigma{:}\varphi$ is the root of the branch. Then, $\mathcal{M}^\Theta, \sigma \models \varphi$.

Ending Comments

- Does the procedure terminate? Yes thanks to the subformula property and definition of saturation (but careful with (Id) rule, see Blackburn at al. for fixing termination for hybrid logic).
- For further discussion, send an e-mail or come at office L303.

D'Agostino, M., Gabbay, D. M., Hähnle, R., & Posegga, J. (Eds.). (1999). *Handbook of Tableaux Methods.*

Blackburn, P., & Bolander, T. (2006). *Termination for Hybrid Tableaux.*

Hoffmann, G. (2010). *Lightweight Hybrid Tableaux.*