Fundamental Notions in Classical Propositional Logic

Soundness and Completeness

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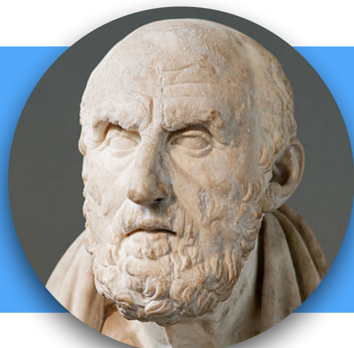
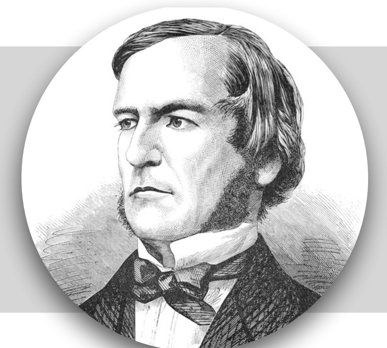
A (Very) Brief History of Propositional Logic



Chrysippus of Soli
(3rd c. BCE)



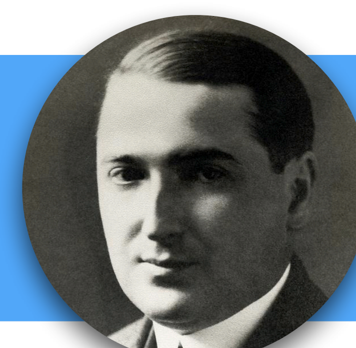
George Boole
(1815–1864)

C. S. Peirce
(1839–1914)

Gottlob Frege
(1848–1925)

David Hilbert
(1862–1943)

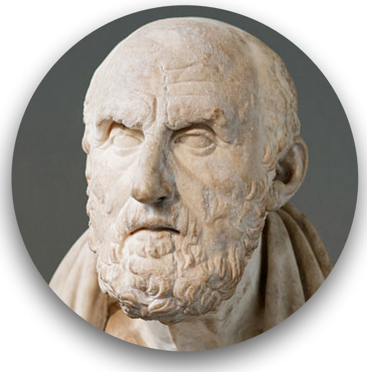
Ludwig Wittgenstein
(1889–1951)



Emil Post
(1897–1954)

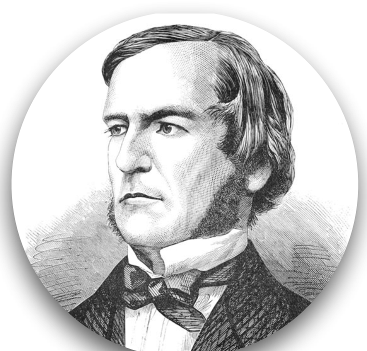
Paul Bernays
(1888–1977)

A (very) Brief History of Propositional Logic



3rd century BCE — Chrysippus of Soli

Founder of Stoic logic; developed the first propositional inference rules (e.g., modus ponens). Treated logic as truth-functional rather than term-based.



Mid-19th century — George Boole (1815–1864)

Boole demonstrated that logical propositions can be expressed as mathematical equations, and that the algebraic manipulation of symbols in those equations offers a method of logical deduction. Provided a formal foundation for symbolic logic.

Late 19th century — Charles Sanders Peirce (1839–1914)

Extended Boolean logic; introduced truth tables and modern logical notation. Unified algebraic and symbolic approaches to logic.

A (very) Brief History of Propositional Logic

Late 19th century — Gottlob Frege (1848–1925)

Introduced formal syntax and semantics (Begriffsschrift, 1879). Established distinction between logical form and content, influencing all later formal logic.

Early 20th century — David Hilbert (1862–1943)

Advanced the formal axiomatic method; sought to prove consistency and completeness of logical systems. His school shaped modern proof theory.

Early 20th century — Ludwig Wittgenstein (1889–1951)

In Tractatus Logico-Philosophicus (1921), described propositions as truth functions; explicitly formulated truth tables and the idea that logic shows the structure of the world.

A (VERY) BRIEF HISTORY OF PROPOSITIONAL LOGIC

1921 — EMIL POST (1897–1954)

Published the first formal proof of completeness for propositional logic. Developed Post systems—abstract models of formal deduction.

1920s–1930s — PAUL BERNAYS (1888–1977)

Collaborated with Hilbert on *Foundations of Mathematics*. Helped formalize proof theory and metalogical results like soundness and completeness theorems.

Soundness and Completeness

The Soundness Theorem states that if a formula can be formally derived within a logical proof system, then it is also true in every possible interpretation. In other words, the axioms and rules of inference never allow us to prove something that is false.

The Completeness Theorem, in contrast, guarantees that if a formula is true in every possible interpretation, then it can also be derived within the formal system. This means that the proof system is sufficiently expressive to capture every logically valid statement. Completeness ensures that no semantically valid truth escapes formal derivation.

Together, these two theorems establish a deep correspondence between syntax (formal proofs) and semantics (truth in models).

Language

Mel Bochner
(1940—2025)

The language of Classical Propositional Logic (CPL) consists of: a denumerable alphabet of proposition symbols, a finite set of logical connectives, and grammar rules that defines how formulas are formed.

The proposition symbols in the alphabet, p, q, r, … each represents a simple statement.

The logical connectives: $\bot$ (falsehood), $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), and $\rightarrow$ (if…then or conditional), enable us to combine proposition symbols into complex formulas.

In BNF form the grammar is given by the rule:

$$\varphi, \psi ::= P \mid \bot \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \rightarrow \psi$$

The grammar specifies that only expressions built according to this rule are well-formed, and thus formulas. In this grammar, we use metavariables to talk about the language itself: symbols like P, Q, R, … refer to propositional symbols, while Greek letters φ, ψ, … indicate arbitrary formulas.

LANGUAGE

Mel Bochner
(1940—2025)

We use **Form** to indicate the set of all formulas of CPL.

The following are some examples of formulas in this set:

1. $p \to (q \to (p \wedge q))$
2. $r \to (\neg q \to \bot\,)$
3. $(\neg p \to r) \to ((r \to p) \vee q)$

The following are examples of expressions not in **Form**:

1. $\wedge \vee qr \bot$ — logical connectives in the wrong place
2. $q \to \top$ — symbol not in the alphabet
3. $(\neg P \to R) \to (\varphi \vee q)$ — use of metavariables

Axiom System

David Hilbert
(1862–1943)

Hilbert's program championed the idea that every scientific theory should be built from explicit axioms and formal rules.

The discovery of *Russell's paradox* in 1902 exposed the limits of existing logical foundations and convinced Hilbert that mathematics required a more rigorous formal framework.

Building on Whitehead and Russell's *Principia Mathematica* (1910–1913), Hilbert began developing a precise logical formalism for axiomatics.

From 1917 onward, working with Paul Bernays and Heinrich Behmann, he advanced formal logic in a series of Göttingen lectures that led to *Principles of Theoretical Logic* (1928).

Bernays's 1918 *Habilitationsschrift*, based on this work, included the first completeness proof for propositional logic, marking a crucial step in the development of modern logic.

Axiom System

David Hilbert
(1862–1943)

The axiom system of CPL is a formal system designed to generate theorems via a combination of axioms and rules of inference.

The axioms express fundamental facts about the interaction of logical connectives. The inference rules specify the permissible steps for deriving new formulas from existing ones.

By repeatedly applying such rule to axioms and previously derived formulas, the system systematically produces theorems, that is, provable statements of the logic.

In this way, the axiom system of CPL serves as a formal engine of logical reasoning. It defines what counts as a valid proof purely in symbolic terms, independent of meaning.

Axiom System

David Hilbert
(1862–1943)

The axioms and rules for CPL are:

Axioms:

(A1) $\bot \to \varphi$

(A2) $\varphi \to (\psi \to \varphi)$

(A3) $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$

(A4) $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$

(A5) $\varphi \to (\psi \to (\varphi \land \psi))$

(A6) $(\varphi \land \psi) \to \varphi$

(A6) $(\varphi \land \psi) \to \psi$

(A7) $\varphi \to (\varphi \lor \psi)$

(A7) $\varphi \to (\psi \lor \varphi)$

(A7) $(\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \lor \psi) \to \chi))$

Rules:

(mp) $\dfrac{\varphi \quad \varphi \to \psi}{\psi}$

AXIOM SYSTEM

David Hilbert
(1862–1943)

A proof in this framework is a finite sequence of formulas, each of which is either an axiom or follows from earlier formulas in the sequence by an application of modus ponens.

Formally, a proof is a sequence $\varphi_1, \ldots, \varphi_n$ of formulas such that, for all $i \in [1 : n]$, we have:

1. φ_i is an instance of an axiom; or
2. there is $j, k \in [1 : i]$ such that $\varphi_k = \varphi_j \to \varphi_i$

The final formula in the sequence is called a theorem.

We write $\Psi \vdash \varphi$, and say that φ is a syntactic consequence of a set of formulas Ψ, iff there is a finite set $\{\psi_1, \ldots, \psi_n\} \subset \Psi$ such that $(\psi_1 \land \ldots \land \psi_n) \to \varphi$ is a theorem.

Axiom System

David Hilbert
(1862–1943)

Below is a proof in the axiom system.

1. $p \to ((q \to p) \to p)$
2. $(p \to ((q \to p) \to p)) \to ((p \to (q \to p)) \to (p \to p))$
3. $(p \to (q \to p)) \to (p \to p)$
4. $p \to (q \to p)$
5. $p \to p$

This proof establishes $p \to p$ is a theorem.

Note also that this proof establishes $\{p\} \vdash p$, that is, p is a syntactic consequence of $\{p\}$.

SEMANTICS

Ludwig Wittgenstein
(1889–1951)

Wittgenstein is largely credited as the first logician to formulate the modern idea of truth tables as a way to represent the logical meaning of formulas.

In his *Tractatus Logico-Philosophicus*, he introduced the idea that such a meaning corresponds to the truth value determined by the constituent elementary propositions under all possible combinations.

To this end, Wittgenstein used tabular diagrams—an innovation that allowed logical connectives to be defined precisely by their truth-functional behavior.

Semantics

Ludwig Wittgenstein
(1889–1951)

The truth tables for the logical connectives of falsum, negation, conjunction, disjunction, and implication, are:

$\bot$
0

φ	$\neg\varphi$
0	1
1	0

φ	ψ	$\varphi \vee \psi$
0	0	0
0	1	1
1	0	1
1	1	1

φ	ψ	$\varphi \wedge \psi$
0	0	0
0	1	0
1	0	0
1	1	1

φ	ψ	$\varphi \to \psi$
0	0	1
0	1	1
1	0	0
1	1	1

SEMANTICS

Ludwig Wittgenstein
(1889–1951)

Formally, a truth table is a matrix representation of a truth function, or an interpretation, that is, a function

$$v: \text{Form} \to \{0,1\}$$

such that

$$v(\bot) = 0$$
$$v(\neg\varphi) = 1 - v(\varphi)$$
$$v(\varphi \land \psi) = \min(v(\varphi), v(\psi))$$
$$v(\varphi \lor \psi) = \max(v(\varphi), v(\psi))$$
$$v(\varphi \to \psi) = \max(1 - v(\varphi), v(\psi))$$

It can be proven that interpretations are uniquely determined by mappings of proposition symbols to truth values. More precisely, any interpretation $v: \text{Form} \to \{0,1\}$ is uniquely determined by a mapping $v': \text{Prop} \to \{0,1\}$, and *viceversa*.

SEMANTICS

Ludwig Wittgenstein
(1889–1951)

We say that a formula φ is satisfiable, and write $\mathsf{SAT}(\varphi)$, iff *exists* an interpretation v such that $v(\varphi) = 1$.

For example, $\neg p \vee q$ is satisfiable.

Moreover, we say that a formula φ is a tautology, written $\mathsf{TAUT}(\varphi)$, iff *for all* interpretations v, we have $v(\varphi) = 1$.

For example, $p \to (q \to p)$ is a tautology.

Finally, we say that a formula φ is a semantic consequence of a set of formulas Ψ, written $\Psi \vDash \varphi$, iff for all interpretations v, if $v(\psi) = 1$ for all $\psi \in \Psi$, then, $v(\varphi) = 1$.

For example, $\{p \to q\} \vDash \neg p \vee q$

Soundness and Completeness

Paul Bernays
(1888–1977)

In their Göttingen lectures, David Hilbert and Paul Bernays worked on various notions of completeness for first-order predicate logic. These lectures led to the *Principles of Theoretical Logic*, which subsequently served as Gödel's main source for his 1929 dissertation.

A key tool developed in these lectures was the method of arithmetical interpretations. In propositional logic, this results in truth-functions.

Using this framework, Hilbert proved both the soundness of the propositional calculus and its Post completeness—that is, adding any non-derivable formula as an axiom would render the system inconsistent.

In his 1918 Habilitationsschrift, Bernays refined Hilbert's approach and established the completeness theorem for the propositional calculus in the now standard form: every valid formula is provable.

Soundness and Completeness

Paul Bernays
(1888–1977)

The **Soundness Theorem** tells us that syntactic consequence implies semantic consequence. Formally, the soundness theorem is

$$\Psi \vdash \varphi \text{ implies } \Psi \vDash \varphi$$

The soundness theorem is proven by induction on the size of a derivation. This proof by induction establishes that all axioms are tautologies, and that the rule of *modus ponens* preserves truth.

This result may not seem very impressive, but it enables us to show that some formulas are not theorems, simply by showing that they are not tautologies. Without this lemma that would have been a very awkward task.

SOUNDNESS AND COMPLETENESS

Paul Bernays
(1888–1977)

In turn, the COMPLETENESS THEOREM tells us that semantic consequence implies syntactic consequence. Formally, the theorem is

$$\Psi \vDash \varphi \text{ implies } \Psi \vdash \varphi$$

This result is much more impressive. It tells us, for instance, that the tedious task of building proofs can be replaced by the (equally tedious, but automatic) task of checking tautologies. This simplifies, at least in theory, the search for theorems considerably; for profs one has to be (moderately) clever, for truth tables one has to possess perseverance.

Soundness and Completeness

Paul Bernays
(1888–1977)

The proof of completeness is carried out by contraposition. Instead of proving $\Psi \vDash \varphi$ implies $\Psi \vdash \varphi$, equivalently, we prove

$$\Psi \nvdash \varphi \text{ implies } \Psi \nvDash \varphi$$

This result is much more impressive. It tells us, for instance, that the tedious task of building proofs can be replaced by the (equally tedious, but automatic) task of checking tautologies. This simplifies, at least in theory, the search for theorems considerably; for profs one has to be (moderately) clever, for truth tables one has to possess perseverance.

Soundness and Completeness

Paul Bernays
(1888–1977)

To prove the completeness theorem, we need a few new notions. The first one has an impressive history; it is the notion of *freedom from contradiction* or *consistency*. It was made the cornerstone of the foundations of mathematics by Hilbert.

We call a set of formulas Ψ consistent iff $\Psi \nvdash \bot$, otherwise we call it inconsistent. In words, a set of formulas is consistent iff it cannot derive a contradiction.

The consistency of a set of formulas has various alternative, equivalent, formulations. For example, we may call Ψ consistent iff exists φ such that $\Psi \nvdash \varphi$; or for no φ, $\Psi \vdash \varphi$ and $\Psi \vdash \neg\varphi$.

Dually, we may say that Ψ is inconsistent iff for all φ, $\Psi \vdash \varphi$; or there exists φ such that $\Psi \vdash \varphi$ and $\Psi \vdash \neg\varphi$.

Soundness and Completeness

Paul Bernays
(1888–1977)

In mathematical practice one tries to establish consistency by exhibiting a model (think of the consistency of the negation of Euclid's fifth postulate and the non-euclidean geometries). For us, consistency means looking for a suitable interpretation.

Lemma. If there is an interpretation v such that, $v(\psi) = 1$ for all $\psi \in \Psi$, then, we have that Ψ is consistent, that is, $\Psi \nvdash \bot$.

Lemma. If $\Psi \nvdash \varphi$, then, $\Psi \cup \{\neg\varphi\}$ is consistent.

Soundness and Completeness

Paul Bernays
(1888–1977)

The following kind of consistent sets play a fundamental role.

We say that a set of formulas Ψ is maximally consistent iff it is consistent, and for any other set of formulas Ψ', if $\Psi \subset \Psi'$, then, Ψ' is inconsistent.

For example, the set $\Psi = \{\psi \mid v(\psi) = 1\}$—for some fixed interpretation v—is maximally consistent.

To see why, note that this set is clearly consistent. Now, choose a set Ψ' such that $\Psi \subset \Psi'$. Then, exists $\psi' \in \Psi'$ such that $\psi' \notin \Psi$. This implies, $v(\psi') = 0$, and so $v(\neg\psi') = 1$, i.e., $\neg\psi' \in \Psi$. The last step tells us $\{\psi, \neg\psi\} \subseteq \Psi'$, which yields Ψ' inconsistent.

Soundness and Completeness

Paul Bernays
(1888–1977)

The next lemma is the logical analogue of the maximal ideal existence lemma from ring theory (or the Boolean prime ideal theorem), and is usually proved via Zorn's lemma.

Lemma. It follows that every consistent set of formulas Ψ is contained in one that is maximally consistent.

The main idea behind the proof of this lemma is to create a list $\varphi_0, \varphi_1, \varphi_2, \ldots$ of all formulas, and based on this list, define:

$$\Psi_0 = \Psi$$

$$\Psi_{(n+1)} = \begin{cases} \Psi_n \cup \{\varphi_n\} & \text{if } \Psi_n \cup \{\varphi_n\} \text{ is consistent} \\ \Psi_n & \text{otherwise} \end{cases}$$

$$\Psi^* = \bigcup_{n \geq 0} \Psi_n$$

SOUNDNESS AND COMPLETENESS

Paul Bernays
(1888–1977)

Maximally consistent sets play a central role in proving completeness, as they provide a systematic way to construct interpretations directly from syntactic information.

To see how this can be done, note that any maximally consistent set Ψ is *complete*, that is, for all φ, either $\varphi \in \Psi$ or $\neg\varphi \in \Psi$.

Further, maximally consistent sets are *closed under provability*, that is $\varphi \wedge \chi \in \Psi$ iff $\varphi, \chi \in \Psi$, $\varphi \vee \chi \in \Psi$ iff $\{\varphi, \chi\} \cap \Psi \neq \emptyset$, and $\varphi \to \chi \in \Psi$ iff $\neg\varphi \vee \chi \in \Psi$.

With these tools at hand, take any maximally consistent set Ψ, we construct a function $v\colon \text{Form} \to \{0,1\}$ such that

$$v(\varphi) = \begin{cases} 1 \text{ if } \varphi \in \Psi \\ 0 \text{ otherwise} \end{cases}$$

It is possible to prove that v is an interpretation.

Soundness and Completeness

Paul Bernays
(1888–1977)

We now have all the tools at hand to sketch out the argument for completeness by contraposition, that is, $\Psi \nvdash \varphi$ implies $\Psi \nvDash \varphi$.

The key steps in this argument are:

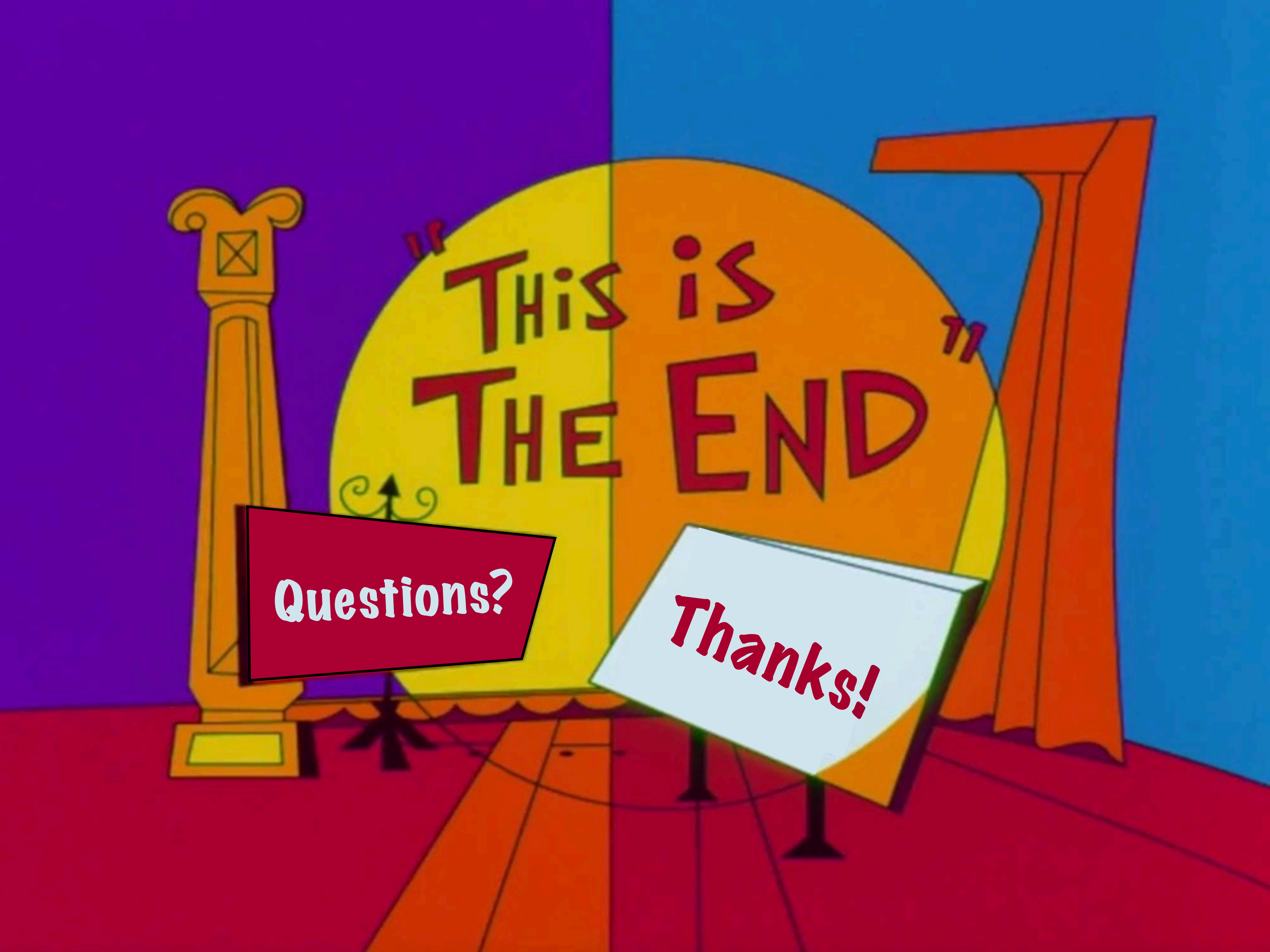
1. Suppose $\Psi \nvdash \varphi$.
2. Then, $\Psi \cup \{\neg\varphi\}$ is consistent.
3. Build a maximally consistent set $\Psi^* \supseteq \Psi \cup \{\neg\varphi\}$.
4. Build the interpretation ν associated to Ψ^*.
5. This interpretation establishes $\Psi \nvDash \varphi$.

Conclusions

The formal language, axiomatic system, and proof methods of Classical Propositional Logic—refined by Hilbert, Bernays, Post, and others—made possible the rigorous study of logical consequence.

The Soundness Theorem ensures that every provable formula is semantically valid, while the Completeness Theorem guarantees that every valid formula is provable. These results establish a perfect correspondence between syntax and semantics.

Historically, these theorems marked a turning point in the foundations of logic and mathematics, linking symbolic reasoning to meaning with exact precision. Their legacy continues in modern proof theory, automated reasoning, and the logical foundations of computer science, where the balance between syntax and semantics remains central.


"This is
The End"
Questions?
Thanks!