Lowness Properties and Approximations of the Jump

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Abstract

We study and compare two combinatorial lowness notions: *strong jump-traceability* and *well-approximability of the jump*, by strengthening the notion of jump-traceability and super-lowness for sets of natural numbers. A computable non-decreasing unbounded function h is called an order function. Informally, a set A is strongly jump-traceable if for each order function h, for each input e one may effectively enumerate a set T_e of possible values for the jump $J^A(e)$, and the number of values enumerated is at most $h(e)$. A' is well-approximable if can be effectively approximated with less than $h(x)$ changes at input x, for each order function h. We prove that there is a strongly jump-traceable set which is not computable, and that if A' is well-approximable then A is strongly jump-traceable. For r.e. sets, the converse holds as well. We characterize jump-traceability and the corresponding strong variant in terms of Kolmogorov complexity, and we investigate other properties of these lowness notions.

1 Introduction

A *lowness property* of a set A says that A is computational weak when used as an oracle, and hence A is close to being computable. In this article we study and compare some "combinatorial" lowness properties in the direction of characterizing K-trivial sets.

A set is K-trivial when it is highly compressible in terms of Kolmogorov complexity (see Section 2 for the formal definition). In [18], Nies proved that a set is K-trivial if and only if A is low for Martin-Löf-random (i.e. each Martin-Löf-random set is already random relative to A).

Terwijn and Zambella [23] defined a set A to be *recursively traceable* if there is a recursive bound p such that for every $f \leq_T A$, there is a recursive r such that for all x, $|D_{r(x)}| \leq p(x)$, and $(D_{r(x)})_{x\in\mathbb{N}}$ is a set of possible values of f: for all x, we have $f(x) \in D_{r(x)}$.

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They showed that this combinatorial notion characterizes the sets that are low for Schnorr tests.

This property was modified in [19] to *jump-traceability.* A set A is jump traceable if its jump at argument e, written $J^A(e) = \{e\}^A(e)$, has few possible values.

Definition 1. A uniformly r.e. family $T = \{T_0, T_1, \ldots\}$ of sets of natural numbers is a *trace* if there is a recursive function h such that $\forall n\ |T_n| \leq h(n)$. We say that h is a *bound* for T. The set A is *jump-traceable* if there is a trace T such that

$$\forall e\ [J^A(e)\downarrow \Rightarrow\ J^A(e) \in T_e].$$

We say that A is jump traceable *via a function* h if, additionally, T has bound h.

Another notion studied in [19] is *super-lowness*, first introduced in [4, 17].

Definition 2. A set A is *ω-r.e.* iff there exists a recursive function b such that $A(x) = \lim_{s\to\infty} g(x,s)$ for a recursive $\{0,1\}$-valued g such that $g(x,s)$ changes at most $b(x)$ times, i.e. $|\{s : g(x,s) \neq g(x,s+1)\}| \leq b(x)$. In this case, we say that A is ω-r.e. *via the function* g and *bound* b. A is *super-low* iff A' is ω-r.e.

Recall that a set A is *low* if $A' \leq_T \emptyset'$. The above definition of A being super-low is equivalent to $A' \leq_{tt} \emptyset'$. Hence super-lowness, clearly implies lowness.

Both jump-traceable and super-low sets are closed downward under Turing reducibility and imply being generalized low (i.e. $A' \leq A \oplus \emptyset'$). In [19] it was proved that these two lowness notions coincide within the r.e. sets but that none of them implies the other within the ω-r.e. sets.

In this article, we define the notions of *strong jump-traceability* and *well-approximability of the jump*, strengthening super-lowness. In the strong variant of these notions consider *all* order functions as the bound instead of just *some* recursive bound. Here, an *order* function is a slowly growing but unbounded recursive function. Our first two results are:

- There is a non-computable strongly jump-traceable set;
- If A' is well-approximable then A is strongly jump-traceable. The converse also holds, if A is r.e.

Our approach is used to study interesting lowness properties related to plain and prefix-free Kolmogorov complexity. We investigate the properties of sets A such that the Kolmogorov complexity relative to A is only a bit smaller than the unrelativized one. We prove some characterizations of jump-traceability and its strong variant in terms of prefix-free (denoted with K) and plain (denoted with C) Kolmogorov complexity, respectively:

- A is jump-traceable if and only if there is a recursive p, growing faster than linearly such that $K(y)$ is bounded by $p(K^A(y) + c_0) + c_1$, for some constants c_0 and c_1;
- A is strongly jump-traceable if and only if $C(x) - C^A(x)$ is bounded by $h(C^A(x))$, for every order function h and almost all x.

Recall that A is low for K iff $K(x) \leq K^A(x) + \mathcal{O}(1)$ for each x. Nies [18] showed that this property is equivalent to being K-trivial. In particular, non-computable low for K sets exist. The corresponding property involving C is only satisfied by the computable sets (because it implies being C-trivial by [7], which is the same as computable). The characterization of strongly jump-traceable is via a property that states that C^A is very close to C, while not implying computability.

We know that K-triviality implies jump-traceability, but it is unknown whether K-triviality implies strong jump-traceability. The reverse direction is also open.

2 Basic definitions

If A is a set of natural numbers then $A(x) = 1$ if $x \in A$; otherwise $A(x) = 0$. We denote by $A \upharpoonright n$ the string of length n which consists of the bits $A(0) \dots A(n-1)$.

If A is given by an effective approximation and Ψ is a functional, we write $\Psi^A(e)[s]$ for $\Psi_s^{A_s}(e)$. From a partial recursive functional Ψ, one can effectively obtain a primitive recursive and strictly increasing function α, called a *reduction function* for Ψ, such that

$$\forall X \, \forall e \; \Psi^X(e) = J^X(\alpha(e)).$$

For each set A, we want to define $K^A(y)$ as the length of a shortest prefix-free description of y using oracle A. An *oracle machine* is a partial recursive functional $M : \{0,1\}^\infty \times \{0,1\}^* \mapsto \{0,1\}^*$. We write $M^A(x)$ for $M(A, x)$. M is an *oracle prefix-free machine* if the domain of M^A is an antichain under inclusion of strings, for each A. Let $(M_d)_{d \in \mathbb{N}}$ be an effective listing of all oracle prefix-free machines. The universal oracle prefix-free machine U is given by

$$U^A(0^d 1 \sigma) = M_d^A(\sigma)$$

and the prefix-free Kolmogorov complexity relative to A is defined as

$$K^A(y) = \min\{|\sigma| : U^A(\sigma) = y\},$$

where $|\sigma|$ denotes the length of σ. If $A = \emptyset$, we simply write $U(\sigma)$ and $K(y)$. As usual, $U(\sigma)[s] \downarrow = y$ indicates that $U(\sigma) = y$ and the computation takes at most s steps. Schnorr's Theorem states that $A \in \{0,1\}^\infty$ is Martin-Löf random iff the initial segments of A have high K-complexity, i.e.

$$\exists c \, \forall n \; K(A \upharpoonright n) > n - c.$$

A set A is K-trivial iff the initial segments of A have low K-complexity, i.e.

$$\exists c \, \forall n \; K(A \upharpoonright n) \leq K(n) + c.$$

We say that $A \leq_K B$ iff

$$\exists c \, \forall n \; K(A \upharpoonright n) \leq K(B \upharpoonright n) + c.$$

The Kraft-Chaitin Theorem states that from a recursive sequence of pairs $(\langle n_i, \sigma_i \rangle)_{i \in \mathbb{N}}$ (known as *requests*) such that $\sum_{i \in \mathbb{N}} 2^{-n_i} \leq 1$, we can effectively obtain a prefix-free machine

M such that for each i there is a τ_i of length n_i with $M(\tau_i)\downarrow=\sigma_i$, and $M(\rho)\uparrow$ unless $\rho=\tau_i$ for some i.

If we drop the condition of the domain of M^A being an antichain, we obtain a similar notion, called plain Kolmogorov complexity and denoted by C. Hence, $C^A(y)$ will denote the length of the shortest description of y using oracle A, when we do not have the restriction on the domain.

A *binary machine* is a partial recursive function $\tilde{M}:\{0,1\}^*\times\{0,1\}^*\mapsto\{0,1\}^*$. Let $\tilde{U}$ be a binary universal function given as

$$\tilde{U}(0^d1\sigma,x)=\tilde{M}_d(\sigma,x),$$

where $(\tilde{M}_d)_{d\in\mathbb{N}}$ is an enumeration of all partial recursive functions of two arguments. We define the plain conditional Kolmogorov complexity $C(y|x)$ as the length of the shortest description of y using $\tilde{U}$ with string x as the second argument, that is,

$$C(y|x)=\min\{|\sigma|:\tilde{U}(\sigma,x)=y\}.$$

Let $str:\mathbb{N}\rightarrow\{0,1\}^*$ be the standard enumeration of the strings. The string $str(n)$ is that binary sequence $b_0b_1\dots b_m$ for which the binary number $1b_0b_1\dots b_m$ has the value $n+1$. Thus, $str(0)=\lambda$, $str(1)=0$, $str(2)=1$, $str(3)=00$, $str(4)=01$ and so on.

3 Strong jump-traceability

Recall that an r.e. set A is *promptly simple* if A is co-infinite and there is a recursive function p and an effective approximation $(A_s)_{s\in\mathbb{N}}$ of A such that, for each e,

$$|W_e|=\infty\ \Rightarrow\ \exists s\ \exists x\ [x\in W_{e,s+1}\setminus W_{e,s}\ \wedge\ x\in A_{p(s)}].$$

In this section, we introduce a stronger version of jump-traceability and we prove that there is a promptly simple (hence non-recursive) strongly jump-traceable set. We also prove that there is no maximal order function as bound for jump-traceability.

Definition 3. A computable function $h:\mathbb{N}\rightarrow\mathbb{N}^+$ is an *order function* if h is non-decreasing and unbounded.

Notice that any reduction function is an order function.

Definition 4. A set A is *strongly jump-traceable* iff for each order function h, A is jump traceable via h.

Clearly, strong jump-traceability implies jump-traceability and it is not difficult to see that strong jump-traceability is closed downward under Turing reducibility.

Proposition 5. *$\{A : A$ is strongly jump-traceable$\}$ is closed downward under Turing reducibility.*

Proof. Suppose A is strongly jump-traceable, $B \leq_T A$. We prove that B is jump-traceable via the given order function h. Let Ψ be the functional such that $\Psi^A(x) = J^B(x)$ for all x and let α be the reduction function such that $J^A(\alpha(x)) = \Psi^A(x)$. We know that A is jump-traceable via a trace $(T_i)_{i\in\mathbb{N}}$ with bound $\tilde{h}$, where $\tilde{h}(z) = h(\min\{y : y \in \mathbb{N} \wedge \alpha(y+1) \geq z\})$. Observe that, since α is an order function, $\tilde{h}$ also is. Clearly,

$$J^B(e) = J^A(\alpha(e)) \downarrow \Rightarrow \; J^B(e) \in T_{\alpha(e)}.$$

Now, $\tilde{h}(\alpha(e)) = h(y)$ for some y such that $\alpha(y) < \alpha(e)$ or $y = 0$. Then $y \leq e$ and $\tilde{h}(\alpha(e)) = h(y) \leq h(e)$. Hence $(T_{\alpha(i)})_{i\in\mathbb{N}}$ is a trace for the jump of B with bound h. □

Clearly each computable set A is strongly jump-traceable, because we can trace the jump by

$$T_e = \begin{cases} \{J^A(e)\} & \text{if } J^A(e)\downarrow; \\ \emptyset & \text{otherwise.} \end{cases}$$

In Theorem 7 below we show the existence of a non-computable strongly jump-traceable set. We need the following result, proven in [16, Theorem 2.3.1]:

Lemma 6. *The function $m(x) = \min\{C(y) : y \geq x\}$ is unbounded, non-decreasing and for every order function f there is an x_0 such that $m(x) < f(x)$ for all $x \geq x_0$. Also, $m(x) = \lim_{s\to\infty} m_s(x)$, where $m_s(x) = m(s,x) = \min\{C_s(y) : s \geq y \geq x \vee y = 0\}$ is recursive and $m_s(x) \geq m_{s+1}(x)$, for all x and s.*

Observe that here $\lambda x, s.C_s(x)$ is the standard recursive approximation from above of $C(x)$ (that is $\lambda s.C_s(x) \to C(x)$ when $s \to \infty$ and $C_s(x) \geq C_{s+1}(x)$).

Theorem 7. *There exist a promptly simple strongly jump-traceable set.*

Proof. We construct a promptly simple set A in stages satisfying the requirements

$$P_e \;:\; |W_e| = \infty \;\Rightarrow\; \exists s \exists x\, [x \in W_{e,s+1} \setminus W_{e,s} \;\wedge\; x \in A_{s+1}].$$

These requirements will ensure that A is promptly simple. Each time we enumerate an element into A in order to satisfy P_e, we may destroy $J^A(k)$, and then our trace for the jump of A will grow. Hence, we must enumerate elements into A in a controlled way, and sometimes we should restrain from putting elements into A. Since for any order function h there has to be a trace for J^A bounded by h, we will work with the function m defined in Lemma 6, which grows slower than any order function. The rule will be that during the construction, P_e may destroy $J^A(k)$ at stage s only if $e < m_s(k)$. (Observe that the restriction on P_e imposed rule may strengthen as s grows, because we may have $m_s(k) > m_{s+1}(k)$.) In this way, we will guarantee that size of our trace for $J^A(e)$ will be bounded by $m(e)$, which will suffice because $m \leq h$ from some point on. As we will see, the exact choice of the trace for J^A with bound h depends on h, and is made in a nonuniform way.

Construction of A. Let m_s be the non-decreasing, unbounded function defined in Lemma 6.

Stage 0: set $A_0 = \emptyset$ and declare P_e unsatisfied for all e.

Stage $s+1$: choose the least $e \leq s$ such that

- P_e yet not satisfied;
- There exists x such that $x \in W_{e,s+1} \setminus W_{e,s}$, $x > 2e$ and for all k such that $m_s(k) \leq e$, if $J^A(k)[s]$ is defined then x is greater than the use of $J^A(k)[s]$.

If such e exists, put least such x for e into A_{s+1}. We say that P_e *receives attention* at stage $s+1$, and declare P_e satisfied. Otherwise, $A_{s+1} = A_s$. Finally, define $A = \bigcup_s A_s$.

Verification. Clearly, P_e receives attention at most once. So we can use below the fact that every requirement influences the enumeration of A at most once.

To show that A is strongly jump-traceable, fix a recursive order function h. We will prove that there exists an r.e. trace T for J^A as in Definition 1. Let h be any order function. By Lemma 6, there exists k_0 such that for all $k \geq k_0$, $m(k) \leq h(k)$. Define the recursive function

$$f(k) = \begin{cases} \min\{s : m_s(k) \leq h(k)\} & \text{if } k \geq k_0; \\ 0 & \text{otherwise.} \end{cases}$$

For $k \geq k_0$ and $s \geq f(k)$, $m_s(k)$ will be below $h(k)$, so $J^A(k)$ may change because P_e receives attention, for $e < m_s(k) \leq h(k)$. Since each P_e receives attention at most once, $J^A(k)$ can change at most $h(k)$ times after stage $f(k)$. So

$$T_k = \begin{cases} \{J^A(k)[s] : J^A(k)[s]\downarrow \ \wedge\ s \geq f(k)\} & \text{if } k \geq k_0; \\ \{J^A(k)\} & \text{if } J^A(k)\downarrow \ \wedge\ k < k_0; \\ \emptyset & \text{otherwise.} \end{cases}$$

is as required.

Fix e such that W_e is infinite and let us see that P_e is met. Let s such that

$$\forall k\ [m(k) \leq e \ \Rightarrow\ m_s(k) = m(k)]$$

and $s' > s$ such that no P_i receives attention after stage s' for any $i < e$. Then, by the construction, no computation $J^A(k)$, $m(k) \leq e$ can be destroyed after stage s'. So there is $t > s'$ such that for all k where $m_t(k) \leq e$, if $J^A(k)$ converges then the computation is stable from stage t on. Choose $t' \geq t$ such that there is $x \in W_{e,t'+1} \setminus W_{e,t'}$, $x > 2e$ and x is greater than the use of all converging $J^A(k)$ for all k where $m_{t'}(k) \leq e$. Now either P_e was already satisfied or P_e receives attention at stage $t'+1$. In either case P_e is met. □

Next we study the size of the trace bound for jump-traceability. Given an order function h, it is always possible to find a jump-traceable set A for which h is too small to be a bound for any trace for the jump of A.

Theorem 8. *For any order function h there is an r.e. set A and an order function $\tilde{h}$ such that A is jump-traceable via $\tilde{h}$ but not via h.*

Proof. We will define an auxiliary functional Ψ and we use α, the reduction function for Ψ (i.e. $\Psi^X(e) = J^X(\alpha(e))$ for all X and e), in advance by the Recursion Theorem. At the same time, we will define an r.e. set A and a trace $\tilde{T}$ for J^A. Finally, we will verify that there is an order function $\tilde{h}$ as stated.

Let $T(0), T(1), \ldots$ be an enumeration of all the traces with bound h, so that

$$T(e) = \{T(e)_0, T(e)_1, \ldots\},$$

the e-th such trace, is as in Definition 1. Requirement P_e tries to show that J^A is not traceable via the trace $T(e)$ with bound h, that is,

$$P_e \quad : \quad \exists x \ \Psi^A(x) \notin T(e)_{\alpha(x)}$$

and requirement N_e tries to stabilize the jump when it becomes defined, i.e.

$$N_e \quad : \quad [\exists^\infty s \ J^A(e)[s] \downarrow] \ \Rightarrow \ J^A(e) \downarrow .$$

The strategy for a single procedure P_e consists of an initial action and a possible later action.

Initial action at stage $\mathbf{s+1}$:

- Choose a new candidate $x_e = \langle e, n\rangle$, where n is the number of times that P_e has been initialized. Define $\Psi^A(x_e)[s+1] = 0$ with large use.

Action at stage $\mathbf{s+1}$:

- Let $x_e = \langle e, n\rangle$ be the current candidate. Put y into A_{s+1}, where y is the use of the defined $\Psi^A(x_e)[s]$. Notice that in the construction this action will not affect $J^A(i)[s]$ for $i < e$ because of the choice of y;
- Define $\Psi^A(x_e)[s+1] = \Psi^A(x_e)[s] + 1$ with use $y' > y$ and greater than the use of all defined computations of $J^A(i)[s+1]$ for $i < e$.

We say that P_e *requires attention* at stage $s+1$ if $\Psi^A(x_e)[s] \in T(e)_{\alpha(x_e)}[s]$ and we say that N_e *requires attention* at stage $s+1$ if $J^A(e)[s]$ becomes defined for the first time.

Construction of A. We define $\tilde{T} = \{\tilde{T}_0, \tilde{T}_1, \ldots\}$ by stages. The s-th stage of $\tilde{T}_i$ will be denoted by $\tilde{T}_i[s]$. We start with $A_0 = \emptyset$ and $\tilde{T}_i[0] = \emptyset$ for all i. At stage $s+1$ we consider the procedures N_j for $j \leq s$ and P_j for $j < s$. We also initialize the new P_s. We look at the least procedure requiring attention in the order

$$P_0, N_0, \ldots, P_s, N_s.$$

If there is no one, do nothing. Otherwise, suppose P_e is the first one. We let P_e take action at $s+1$, changing A below the use of $\Psi^A(x_e)[s]$ and redefining $\Psi^A(x_e)[s+1]$ without affecting N_i for $i < e$. We keep the other computations of P_j with the new definition of A, for $j \neq i$ and large use. If N_e is the least procedure requiring attention, there is y such that $J^A(e)[s] \downarrow= y$. We put y into $\tilde{T}_e[s+1]$ and initialize P_j for $e < j \leq s$. In this case, we say that N_e *acts*.

Verification. Let us prove that P_e is met. Take s such that all $J^A(i)$ are stable for $i < e$. Suppose x_e is the actual candidate of P_e. Since P_e is not going to be initialized again, x_e is the last candidate it picks. Each time $\Psi^A(x_e)[t] \in T(e)_{\alpha(x_e)}[t]$ for $t > s$, P_e acts and changes the definition of $\Psi^A(x_e)$ to escape from $T(e)_{\alpha(x_e)}$. Since $|T(e)_{\alpha(x_e)}| \leq h(\alpha(x_e))$, there is $s' > s$ such that $T(e)_{\alpha(x_e)}[s'] = T(e)_{\alpha(x_e)}$. By construction, $\Psi^A(x_e)[s'+1] \notin T(e)_{\alpha(x_e)}$ and $\Psi^A(x_e)[s'+1]$ is stable.

We say that N_e is *injured* at stage $s+1$ if we put y into A_{s+1} and y is less or equal than the use of $J^A(e)[s]$. We define $c_P(k)$ as a bound for the number of initializations of P_r, for $r \leq k$; and define $c_N(k)$ as a bound for the number of injuries to N_r, for $r \leq k$. Since P_0 is initialized just once and makes at most $h(\langle 0,0\rangle)$ changes in A, $c_P(0) = 1$ and $c_N(0) = h(\langle 0,0\rangle)$. The number of times that P_{k+1} is initialized is bounded by the number of times that N_r acts, for $r \leq k$, so

$$c_P(k+1) = c_P(k) + c_N(k).$$

Each time N_r is injured, for $r \leq k$ then N_{k+1} may also be injured; additionally, N_{k+1} may be injured each time P_{k+1} changes A. The latter occurs at most $h(\langle k+1, i\rangle)$ for the i-th initialization of P_{k+1}. Hence

$$c_N(k+1) = 2c_N(k) + \sum_{i \leq c_P(k+1)} h(\langle k+1, i\rangle).$$

Once N_e is not injured anymore, if $J^A(e)\downarrow$ then $J^A(e) \in \tilde{T}_e$. Since the number of changes of $J^A(k)$ is at most the number of injuries to N_e, we define the function $\tilde{h}(e) = c_N(e)$ which is clearly an order function and it constitutes a bound for the trace $(\tilde{T}_i)_{i\in\mathbb{N}}$. □

It is open if there is *minimal* bound for jump-traceability. That is, given an order function h, is there a set A and an order function $\tilde{h}$ such that A is jump-traceable via h but not via $\tilde{h}$. If this fails for some order function h, then strong jump traceability is the same as jump traceability for that single order function.

4 Well-approximability of the jump

We strengthen the notion of super-lowness and study the relationship to strongly jump-traceability.

Definition 9. A set D is *well-approximable* iff for each order function b, D is ω-r.e. via b.

Clearly, if A' is well-approximable, then A is super low. It is not difficult to see that well-approximability of the jump is closed downward under Turing reducibility.

Proposition 10. *$\{A : A'$ is well approximable$\}$ is closed downward under Turing reducibility.*

Proof. Suppose A is such that A' is well-approximable, and let $B \leq_T A$. We prove that B' is well-approximable via the given order function b. Define Ψ and α as in Proposition 5. We know that there is a recursive $\{0,1\}$-valued g such that $A'(x) = \lim_{s\to\infty} g(x,s)$ and $g(x,s)$ changes at most $\tilde{b}(x)$ times, where $\tilde{b}(z) = b(\min\{y : y \in \mathbb{N} \wedge \alpha(y+1) \geq z\})$. Then

$$\lim_{s\to\infty} g(\alpha(x), s) = A'(\alpha(x)) = B'(x)$$

and $g(\alpha(x), s)$ changes at most $\tilde{b}(\alpha(x))$ times. As in Proposition 5, $\tilde{b}(\alpha(x)) \leq b(x)$. □

We next prove that if A is r.e. then A is strongly jump-traceable iff A' is well-approximable. We first need the following lemmas.

Lemma 11. *Let f and $\hat{f}$ be order functions such that $f(x) \leq \hat{f}(x)$ for almost all x.*

(i) *If A is jump-traceable via f then A is jump traceable via $\hat{f}$;*

(ii) *If A is well-approximable via f then A is well-approximable via $\hat{f}$.*

Proof. Assume

$$\exists x_0\ \forall x\ [x \geq x_0 \ \Rightarrow\ f(x) \leq \hat{f}(x)].$$

For (i), suppose T is a trace for J^A with bound f. We can define the trace $\hat{T}$:

$$\hat{T}_x = \begin{cases} T_x & \text{if } x \geq x_0; \\ \{J^A(x)\} & \text{otherwise.} \end{cases}$$

Hence, if $x \geq x_0$ then $|\hat{T}_x| = |T_x| \leq f(x) \leq \hat{f}(x)$, and if $x < x_0$ then $1 = |\hat{T}_x| \leq \hat{f}(x)$.

For (ii), suppose A is well-approximable via the $\{0,1\}$-valued $g(x,s)$ which changes at most $f(x)$ times. Define

$$\hat{g}(x,s) = \begin{cases} g(x,s) & \text{if } x \geq x_0; \\ A(x) & \text{otherwise.} \end{cases}$$

If $x \geq x_0$ then $\hat{g}(x,s)$ changes at most $f(x) \leq \hat{f}(x)$ times, and if $x < x_0$ then $\hat{g}$ does not change at all. □

Lemma 12. *There exists a recursive γ such that for all r.e. A:*

(i) *If A is jump-traceable via an order function h then A is super-low via the order function $b(x) = 2h(\gamma(x)) + 2$;*

(ii) *If A is super-low via an order function b then A is jump-traceable via the order function $h(x) = \lfloor \frac{1}{2} b(\gamma(x)) \rfloor$.*

Proof. We follow the proof of [19, Theorem 4.1], together with Lemma 11.

(i)⇒(ii). Suppose A is jump-traceable via h. By [19] A is super-low via a $\{0, 1\}$-valued recursive g such that $g(x, s)$ changes at most $2h(\alpha(x))+2$ times. Here, α is a reduction function (hence primitive recursive) which depends on A. The diagonal γ of the Ackermann-function satisfies $\gamma(x) \geq \alpha(x)$ for almost all x [20, Volume 2, Theorem VIII.8.10]. Since h is an order function, $2(h \circ \gamma) + 2$ also is, and $2h(\gamma(x)) + 2 \geq 2h(\alpha(x)) + 2$ for almost all x. By Lemma 11, A is super-low via $b(x) = 2h(\gamma(x)) + 2$.

(ii)⇒(i). Suppose A is super-low via an order function b and the $\{0, 1\}$-valued function g. Again following [19], there is a trace for J^A via $\lfloor \frac{1}{2}(b \circ \gamma) \rfloor$, for a primitive recursive α which depends on g. As we did in the previous implication, $\lfloor \frac{1}{2} b(\gamma(x)) \rfloor \geq \lfloor \frac{1}{2} b(\alpha(x)) \rfloor$ for almost all x. Thus A is jump-traceable via $h(x) = \lfloor \frac{1}{2} b(\gamma(x)) \rfloor$. □

Theorem 13. *Let A be an r.e. set. Then the following are equivalent:*

(i) *A is strongly jump-traceable;*

(ii) *A' is well-approximable.*

Proof. (i)⇒(ii). Given an order function b, let us prove that A is super-low via b. By part (i) of Lemma 12, it suffices to define an order function h such that $2h(\gamma(x)) + 2 \leq b(x)$ for almost all x. If $b(x) \geq 4$ then define $h(\gamma(x)) = \lfloor \frac{b(x)-2}{2} \rfloor$ and if $b(x) < 4$, define $h(\gamma(x)) = 1$. Since γ can be taken strictly monotone, the above definition is correct and we can complete it to make h an order function.

(ii)⇒(i). Given an order function h, we will prove that A is jump-traceable via h. By part (ii) of Lemma 12, it suffices to define an order function b such that $\lfloor \frac{1}{2} b(\gamma(x)) \rfloor \leq h(x)$ for almost all x. The argument is similar to the previous case. □

Later, in Corollary 18, we will improve this result and we will see that, in fact, the implication (ii)⇒(i) holds for any A.

We finish this section by proving that the prefixes $D \upharpoonright n$ of a well-approximable set D have low Kolmogorov complexity, of order logarithmic in n. Hence D is not Martin-Löf random and furthermore, the effective Hausdorff dimension is 0. The latter is just equivalent of saying that there is no $c > 0$ such that cn is a linear lower bound for the prefix-free Kolmogorov complexity of $D \upharpoonright n$ for almost all n.

Theorem 14. *If D is well-approximable then for almost all n, $K(D \upharpoonright n) \leq 4|n|$.*

Proof. Suppose $D(n) = \lim_{s \to \infty} g(n, s)$, where g is recursive and changes at most n times. Given n, there is a unique s and some $m < n$ such that $g(m, s) \neq g(m + 1, s)$ but $g(q, t) = g(q, t + 1)$ for all $t > s$ and $q < n$. That is, s is the time when g converges on below n and m is the place where the last change takes place. The stage s can be computed from m and the number k of stages with $g(m, t+1) \neq g(m, t)$. So one can compute $D \upharpoonright n$ from m, n, k. Since $k, m \leq n$, one can, for almost all n, code m, n, k in a prefix-free way in $4|n|$ many bits. This is done by using a prefix of the form 1^q0 followed by $2q$ bits representing n, $2q$ bits representing m and $2q$ bits representing k as binary numbers; here q is just the smallest number such that $2q$ bits are enough. Since $k, m \leq n$ and since $2q \leq |n| + c$ for some constant c and since the additionally necessary coding needed to transform the above representation into a program for U is bounded by a constant, we have that there is a constant d such that

$$\forall n\ K(D \upharpoonright n) \leq 3|n| + |n|/2 + d$$

and then the relation $K(D \upharpoonright n) \leq 4|n|$ holds for almost all n. In fact, using binary notation to store q instead of 1^q0, it would even give

$$K(D \upharpoonright n) \leq 3(|n| + \log(|n|))$$

for almost all n. □

5 Traceability and plain Kolmogorov complexity

We give a characterization of strong jump-traceability in terms of plain Kolmogorov complexity and we show that if A' is well-approximable then A is strongly jump-traceable for any set A.

Theorem 15. *If A' is well-approximable then for every order function h and almost all x, $C(x) \leq C^A(x) + h(C^A(x))$.*

Proof. The idea of the proof is the following. Let h be any order function. Suppose q_x is a minimal A-program for x. We know that there is a c such that $C(x) \leq |q_x| + 2C(x|q_x) + c$. Since $|q_x| = C^A(x)$, we only need to show that $2C(x|q_x) + c \leq h(|q_x|)$ for almost all x. Given q_x and the value of $C(x|q_x)$, we can find a program p_x of length $C(x|q_x)$ which describes x with the help of q_x, that is $\tilde{U}(p_x, q_x) = x$. It can be shown that there is a recursive $\{0,1\}$-valued approximation of the bits of p_x which changes few times (in the proof, this is done with the help of the functional Ψ). Hence, x can be described by the values of $C(x|q_x)$, q_x and p_x. We can represent p_x with the number of changes of the mentioned $\{0,1\}$-valued approximation. This will show $C(x|q_x) \leq 2|h(|q_x|)| + \mathcal{O}(1)$, which is sufficient to get the desired upper bound on $2C(x|q_x) + c$.

Here are the details. Let $\Psi^A(m, n, q)$ be a functional which does the following:

(i) Compute $x = U^A(q)$. If $U^A(q) \uparrow$ then $\Psi^A(m, n, q) \uparrow$;

(ii) Find the first program p such that $|p| = n$ and $\tilde{U}(p, q) = x$. If there is no such p then $\Psi^A(m, n, q) \uparrow$;

(iii) In case $m \notin [1, n]$ then $\Psi^A(m, n, q) \uparrow$. Otherwise, if the m-th bit of p is 1 then $\Psi^A(m, n, q) \downarrow$, else $\Psi^A(m, n, q) \uparrow$.

Let α be a reduction function such that $J^A(\alpha(m, n, q)) = \Psi^A(m, n, q)$. Choose an order function b such that $b(\alpha(n, n, q)) \leq nh(|q|)$ for all n, q. We can approximate $A'(x)$ with a $\{0, 1\}$-valued recursive function which changes at most $b(x)$ times.

Let q_x be a minimal A-program for x, that is, $U^A(q_x) = x$ and $|q_x| = C^A(x)$. Let $n_x = C(x|q_x)$. Then $\Psi^A(m, n_x, q_x) \downarrow$ iff the m-th bit of p_x is 1, where p_x is the first program such that $|p_x| = n_x$ and $\tilde{U}(p_x, q_x) = x$.

Since A' is ω-r.e. via b,

$$p_x = A'(\alpha(1, n_x, q_x)) \ldots A'(\alpha(n_x, n_x, q_x))$$

changes at most

$$\begin{aligned} n_x \max\{b(\alpha(m, n_x, q_x)) : 1 \leq m \leq n_x\} &\leq n_x b(\alpha(n_x, n_x, q_x)) \\ &\leq n_x^2 h(|q_x|) \end{aligned}$$

many times. Since $\tilde{U}(p_x, q_x) = x$ and we can describe p_x with n_x, q_x and the number of changes of $A'(\alpha(1, n_x, q_x)) \ldots A'(\alpha(n_x, n_x, q_x))$, we have

$$\begin{aligned} n_x = C(x \mid q_x) &\leq 2|n_x| + |n_x^2 h(|q_x|)| + \mathcal{O}(1) \\ &\leq 4|n_x| + |h(|q_x|)| + \mathcal{O}(1). \end{aligned} \tag{1}$$

To finish, let us prove that for almost all x, $n_x \leq 2|h(|q_x|)| + \mathcal{O}(1)$. Since $C(x) \leq |q_x| + 2n_x + \mathcal{O}(1)$, this upper bound of n_x will imply that

$$\begin{aligned} C(x) &\leq |q_x| + h(|q_x|) \\ &= C^A(x) + h(C^A(x)) \end{aligned}$$

for almost all x, as we wanted. Hence, let us see that $n_x \leq 2|h(|q_x|)| + \mathcal{O}(1)$ for almost all x. There is a constant N such that for all $n \geq N$, $8|n| \leq n$. We know that for almost all x, q_x satisfies $|h(|q_x|)| \geq N$. Suppose x has this property. Then either $n_x \leq |h(|q_x|)|$ or $4|n_x| \leq n_x/2$. In the second case $n_x - 4|n_x| \geq n_x/2$ and by (1), $n_x/2 \leq |h(|q_x|)| + \mathcal{O}(1)$. So, in both cases, we have $n_x \leq 2|h(|q_x|)| + \mathcal{O}(1)$. □

Lemma 16. *For all $x \in \{0, 1\}^*$ and $d \in \mathbb{N}$,*

$$|\{y : C(x, y) \leq C(x) + d\}| \leq \mathcal{O}(d^4 2^d).$$

Proof. Chaitin [6] proved that

$$\forall d, n \in \mathbb{N}\ |\{\sigma : |\sigma| = n\ \wedge\ C(\sigma) \leq C(n) + d\}| \leq \mathcal{O}(2^d).$$

Let c be such that $\forall x\ C(x) \leq str^{-1}(x) + c$. Consider the partial recursive function $f(x, y, d)$ which enumerates all strings z such that $C(z) \leq str^{-1}(x) + d + c$ until it finds $z = y$. If z

was the i-th string to appear in the enumeration, then $f(x, y, d)$ is the number i written in binary with initial zeroes such that $|f(x, y, d)| = str^{-1}(x) + d + c + 1$. Notice that it is always possible to write $f(x, y, d)$ in this way because there are at most $2^{str^{-1}(x)+d+c+1}$ such strings z. If no such z exists, then $f(x, y, d) \uparrow$. Let x and d be given. Consider y such that $C(x, y) \leq C(x) + d$. Since $C(x, y) \leq str^{-1}(x) + d + c$ then $f(x, y, d) \downarrow$ and

$$\begin{aligned} C(f(x,y,d)) &\leq C(x,y) + 2|d| + \mathcal{O}(1) \\ &\leq C(x) + d + 2|d| + \mathcal{O}(1) \\ &\leq C(str^{-1}(x) + d + c + 1) + d + 4|d| + \mathcal{O}(1). \end{aligned}$$

The last inequality holds because we can compute the string x from the numbers $str^{-1}(x) + d + c + 1$ and d. Let $n = str^{-1}(x) + d + c + 1$ and $d' = d + 4|d| + \mathcal{O}(1)$. For fixed x and d, the mapping $y \mapsto f(x, y, d)$ is injective and thus

$$\begin{aligned} |\{y : C(x,y) \leq C(x) + d\}| &\leq |\{\sigma : |\sigma| = n \ \wedge \ C(\sigma) \leq C(n) + d'\}| \\ &\leq \mathcal{O}(2^{d'}) = \mathcal{O}(d^4 2^d). \end{aligned}$$

This completes the proof. □

Theorem 17. *The following are equivalent:*

(i) *A is strongly jump-traceable;*

(ii) *For every order function h and almost every x, $C(x) \leq C^A(x) + h(C^A(x))$.*

Proof. For any function f, let $\hat{f}(y) = y + f(y)$ for all y.

(i)⇒(ii). Let h_0 be a given order function. It is sufficient to show that $C(x) \leq \hat{h}(C^A(x)) + \mathcal{O}(1)$ for almost all x, where $h = \lfloor h_0/2 \rfloor$. Let α be a reduction function such that $J^A(\alpha(x)) = U^A(str(x))$. Let T be a trace for J^A with bound g such that $g(\alpha(x)) \leq h(|str(x)|)$. Let $m \in \mathbb{N}$ be such that $U^A(str(m)) = y$ and $|str(m)| = C^A(y)$. Since $y \in T_{\alpha(m)}$, we can code y with m and a number not greater than $g(\alpha(m))$ (representing the time in which y is enumerated into $T_{\alpha(m)}$), using at most

$$|str(m)| + g(\alpha(m)) \leq C^A(y) + h(C^A(y))$$

many bits. Then $\forall y\ C(y) \leq \hat{h}(C^A(y)) + \mathcal{O}(1)$.

(ii)⇒(i). Since there are at most $2^n - 1$ programs of length $< n$, $\forall n\ \exists x\ [|x| = n \wedge\ n \leq C(x)]$. Let c be a constant such that

$$\forall x\ [J^A(|x|) \downarrow \Rightarrow\ C^A(x, J^A(|x|)) \leq |x| + c].$$

This last inequality holds because, given x, we can compute $J^A(|x|)$ relative to A.

Let h be any order function and let us prove that A is jump-traceable via h. Define the order function g such that for almost all e, $3^{g(e+c)} \leq h(e)$. By hypothesis, for almost all x, if $J^A(|x|)\downarrow$ then

$$\begin{aligned} C(x, J^A(|x|)) &\leq \hat{g}(C^A(x, J^A(|x|))) \\ &\leq |x| + g(|x| + c) + c. \end{aligned}$$

Define the trace

$$T_e = \{y : \forall x\ [|x| = e \ \Rightarrow\ C(x, y) \leq e + g(e + c) + c]\}.$$

It is clear that for almost all e, if $J^A(e)\downarrow$ then $J^A(e) \in T_e$, because given x such that $|x| = e$, we have $C(x, J^A(e)) \leq e + g(e + c) + c$. To verify that for almost all e, $|T_e| \leq h(e)$, suppose $y \in T_e$. Take x, $|x| = e$ and $C(x) \geq e$. Then

$$\begin{aligned} C(x, y) &\leq e + g(e + c) + c \\ &\leq C(x) + g(e + c) + c. \end{aligned}$$

By Lemma 16, for almost all e there are at most $3^{g(e+c)} \leq h(e)$ such y's in T_e. □

In [19], it was proven that there is a super-low which is not jump-traceable (namely, a super-low Martin-Löf random set). In contrast, from Theorem 15 and Theorem 17 we can conclude that the strong version of super-lowness implies strong jump-traceability.

Corollary 18. *If A' is well-approximable then A is strongly jump-traceable.*

6 Variations on K-triviality

Throughout this section, let $p : \mathbb{N} \to \mathbb{N}$ be strictly increasing such that in addition $\lim_n p(n) - n = \infty$. We call p an *estimation function* if, in addition, $p(n) = \lim_s p_s(n)$ where $p_{s+1}(n) \leq p_s(n)$, and the function $\lambda s, n.p_s(n)$ is recursive. An example of such a function is $q(n) = n + 5 \cdot \min\{K(m) : m \geq n\}$ with the approximation $q_s(n) = n + 5 \cdot \min\{K_s(m) : s \geq m \geq n\}$. Recall that A is K-trivial iff

$$\exists c\ \forall n\ K(A \upharpoonright n) \leq K(n) + c.$$

Nies [18] showed that A is K-trivial if and only if A is low for K, i.e. $\exists c\ \forall x\ K(x) \leq K^A(x) + c$. In this section we weaken the notion of lowness for K:

Definition 19. (i) A set A is *weakly p-low* iff $\forall n\ K(A \upharpoonright n) \leq p(K(n) + c_0) + c_1$ for some constants c_0 and c_1. Let $\mathcal{K}[p]$ denote the class of such sets.

(ii) A set A is *p-low* iff $\forall y\ K(y) \leq p(K^A(y) + c_0) + c_1$ for some constants c_0 and c_1. Let $\mathcal{M}[p]$ denote the class of such sets.

Proposition 20. (i) *If $A \in \mathcal{M}[p]$ and $B \leq_T A$, then $B \in \mathcal{M}[p]$.*

(ii) *If $A \in \mathcal{K}[p]$ and $B \leq_K A$ or $B \leq_{wtt} A$, then $B \in \mathcal{K}[p]$.*

(iii) *Suppose p is an estimation function. Then no random set is in $\mathcal{K}[p]$.*

(iv) *If $A, B \in \mathcal{K}[p]$ and A, B are r.e., then*

$$A \oplus B = \{2x : x \in A\} \cup \{2x+1 : x \in B\} \in \mathcal{K}[p].$$

(v) *$\mathcal{M}[p] \subseteq \mathcal{K}[p]$.*

Proof. (i). Since $B \leq_T A$, there exists a constant c_2 such that for each string y, $K^A(y) \leq K^B(y) + c_2$. Then

$$\begin{aligned} K(y) &\leq p(K^A(y) + c_0) + c_1 \\ &\leq p(K^B(y) + c_0 + c_2) + c_1. \end{aligned}$$

(ii). This is trivial for $\leq_K$. Now suppose $B = \Gamma^A$ for a weak truth-table reduction Γ with recursive bound f. Without loss of generality, we may assume f strictly increasing. Given $A \upharpoonright f(n)$ we can compute n and $B \upharpoonright n$, and then there is a constant c_2 such that for all n,

$$\begin{aligned} K(B \upharpoonright n) &\leq K(A \upharpoonright f(n)) + c_2 \\ &\leq p(K(f(n)) + c_0) + c_1 + c_2. \end{aligned}$$

Since f is recursive, we have $K(f(n)) \leq K(n) + \mathcal{O}(1)$, and hence $B \in \mathcal{K}[p]$.

(iii). Assume $\forall n\ K(A \upharpoonright n) > n - c$ and $A \in \mathcal{K}[p]$ via constants c_0 and c_1. Define the strictly increasing recursive function $\tilde{p}(0) = p_0(0)$ and $\tilde{p}(k+1) = p_0(j)$, where $j = \min\{i : i > k \ \wedge\ p_0(i) > \tilde{p}(k)\}$. Since $\tilde{p} \geq p$, $A \in \mathcal{K}[\tilde{p}]$. Define the Kraft-Chaitin set $\{\langle i, n_i \rangle : i \in \mathbb{N}^+ \ \wedge\ n_i = \tilde{p}(i + d + c_0) + c_1 + c\}$ for M_d with d given in advance by the Recursion Theorem. Then $K(n_i) \leq i + d$ and hence $\tilde{p}(K(n_i) + c_0) \leq \tilde{p}(i + d + c_0)$. Finally,

$$\begin{aligned} K(A \upharpoonright n_i) &\leq \tilde{p}(K(n_i) + c_0) + c_1 \\ &\leq \tilde{p}(i + d + c_0) + c_1 = n_i - c, \end{aligned}$$

and this is a contradiction.

(iv). Ignoring constants, for each n,

$$\begin{aligned} K(A \oplus B \upharpoonright n) &\leq K(A \oplus B \upharpoonright 2n) \\ &\leq \max\{K(A \upharpoonright n), K(B \upharpoonright n)\} \\ &\leq p(K(n)). \end{aligned}$$

In the second inequality we used [12, Theorem 6.4].

(v). Again ignoring constants, for all n,

$$\begin{aligned} K(A \upharpoonright n) &\leq p(K^A(A \upharpoonright n)) \\ &\leq p(K^A(n)) \\ &\leq p(K(n)). \end{aligned}$$

This completes the proof. □

The following proposition shows a connection between jump-traceability and p-lowness. In Theorem 17 we proved a similar result, relating strong jump-traceability and plain Kolmogorov complexity.

Proposition 21. (i) *Suppose p is a recursive function. There is a constant c such that if $A \in \mathcal{M}[p]$ via constants c_0 and c_1 then A is jump-traceable via $h(x) = 2^{p(2|x|+c_0+c)+c_1+1}$;*

(ii) *There is a reduction function α such that if A is jump-traceable via h then $A \in \mathcal{M}[p]$ for $p(z) = 3z + 2|h(\alpha(2^{z+1}))|$.*

Proof. For (i), we know that there is a constant c such that $K^A(J^A(x)) \leq 2|x| + c$ because we can compute $J^A(x)$ from x and the oracle A. Define the trace

$$T_x = \{U(\sigma) : |\sigma| \leq p(2|x| + c_0 + c) + c_1\}.$$

Clearly $|T_x| \leq 2^{p(2|x|+c_0+c)+c_1+1}$. Let $y = J^A(x)$. By hypothesis $K(y) \leq p(K^A(y) + c_0) + c_1$ and then $K(y) \leq p(2|x| + c + c_0) + c_1$. Hence $y \in T_x$.

For (ii), let α be a reduction function such that $J^A(\alpha(x)) = U^A(str(x))$. Let T be a trace for J^A with bound h and let us define the trace

$$\tilde{T}_n = \bigcup_{x:|str(x)|=n} T_{\alpha(x)}.$$

Notice that

$$\begin{aligned} |\tilde{T}_n| &\leq \sum_{x:|str(x)|=n} h(\alpha(x)) \\ &\leq 2^n h(\alpha(2^{n+1})), \end{aligned}$$

since α is increasing. Let $m \in \mathbb{N}$ be such that $U^A(str(m)) = y$ and $|str(m)| = K^A(y)$. Since $y \in T_{\alpha(m)}$, we know that $y \in \tilde{T}_{|str(m)|}$, hence we describe y by saying "y is the i-th element enumerated into $\tilde{T}_{|str(m)|}$". If we code $|str(m)|$ in unary and we code i with

$$\begin{aligned} 2|i| &\leq 2|2^{|str(m)|} h(\alpha(2^{|str(m)|+1}))| \\ &\leq 2|str(m)| + 2|h(\alpha(2^{|str(m)|+1}))| \end{aligned}$$

many bits, we have $K(y) \leq p(K^A(y)) + \mathcal{O}(1)$, for $p(z) = 3z + 2|h(\alpha(2^{z+1}))|$. □

Corollary 22. *A is jump-traceable iff there exists a recursive function p (of the type considered in this section) such that* $A \in \mathcal{M}[p]$.

Figueira, Stephan and Wu [14, Proposition 6] used a universal machine which has the property that there is an approximation K_s of K from above with $K_x(x) = K(x)$ for all $x \in X$ where $X = \{x : \forall y > x\,(K(y) > K(x))\}$. For the following example, such a universal machine is assumed. The next example shows that there is a set in $\mathcal{M}[q]$ where q is as defined at the beginning of Section 6 which is not K-trivial. Note that r differs from the function in Lemma 6 only by using K instead of C and has the same properties as the function given there.

Example 23. *Let* $r(n) = \min\{K(m) : m \geq n\}$ *and* $q(n) = n + 5 \cdot r(n)$. *Then there is a set* $A \in \mathcal{M}[q] \setminus \Delta^0_2$.

Proof. Note that the set $X = \{x : \forall y > x\,\forall t\,(K_t(y) > K_x(x))\}$ is co-r.e. and that it has a co-r.e. subset Y of the form $\{y_0, y_1, \ldots\}$ such that, for all n, $y_n = K(y_{n+1}) = K_{y_{n+1}}(y_{n+1})$. As $K(0) > 0$ one might have the undesirable property that $y_{n+1} < y_n$ for some n. But as there are only finitely many numbers x with $K(x) > x$, one simply adds to the construction of Y the condition that y_0 is taken to be the first element of X larger than these finitely many exceptions and so one has the additional property that $y_{n+1} > y_n$ for all n.

Now one defines a partition $I_0, I_1, \ldots$ of the natural numbers into intervals such that $|I_x| = K_x(K_x(x))$ and $\max(I_x) + 1 = \min(I_{x+1})$. Note that none of these intervals is empty as $K_x(K_x(x)) > 0$ for all x which is due to the fact that a prefix-free universal machine is undefined on the empty input.

Having the partition, one defines a partial-recursive function ψ in stages s where one does the following algorithm where ψ is everywhere undefined before stage 0. The set E will be chosen such that its characteristic function is a suitable extension of ψ and let ψ_s denote the approximation to ψ before stage s.

- Find the least x, y such that $x \leq s$, $y \in I_x$, $\psi(y)$ is undefined and either (1) $x \notin Y_s$ or (2) there is a string $\sigma \in \{0,1\}^{\max(I_x)+1}$ such that $K_s(\sigma) < K_s(x) + 0.5 \cdot \log(|I_x|)$ and σ is consistent with ψ_s, that is, $\psi_s(z) = \sigma(z)$ for all $z \in \mathrm{domain}(\psi_s) \cap \{0, 1, \ldots \max(I_x)\}$.
- In the case that no x, y were found, let $\psi_{s+1} = \psi_s$.
- In the case that x, y were found according to condition (1), let $\psi_{s+1}(y) = 0$ and let $\psi_{s+1}(z) = \psi_s(z)$ for all $z \neq y$.
- In the case that x, y were found according to condition (2), let $\psi_{s+1}(y) = 1 - \sigma(y)$ and let $\psi_{s+1}(z) = \psi_s(z)$ for all $z \neq y$.

Now let A be a set whose characteristic function extends ψ and which is low for Ω. Such a set A exists since ψ defines a Π^0_1 class and Downey, Hirschfeldt, Miller and Nies [11] showed every Π^0_1 class (of sets) has a member which is low for Ω.

Reviewing the construction of ψ, condition (1) enforces that ψ is defined on the complete interval I_x if $x \notin Y$ and condition (2) enforces that if $x = y_n$ and n is large enough then the

Kolmogorov complexity of $A \upharpoonright \max(I_{y_n})$ is at least $K(y_n) + \log(|I_{y_n}|)/2$. To see this, one should have in mind that $x \to \max(I_x)$ is a recursive injective function, that $K_{y_n}(y_n) = K(y_n)$ and that the number of σ of length $\max(I_{y_n}) + 1$ with $K(\sigma) \leq K(y_n) + \log(|I_{y_n}|)/2$ is bounded by a function proportional to $\sqrt{|I_{y_n}|}$. So there will for all sufficiently large n remain some elements in I_{y_n} where ψ is undefined. As the intervals I_{y_n} are of unbounded length, this enforces that for sufficiently large n the value of $K(A \upharpoonright \max(I_{y_n}))$ is at least $K(y_n)+\log(|I_{y_n}|)/2$ while on the other hand $K(\max(I_{y_n}))$ is only a constant above $K(y_n)$. So A is not K-trivial. Since every low for Ω set is either K-trivial or not Δ^0_2, A is also not Δ^0_2, that is, not limit-recursive.

Now it is shown that the set A constructed satisfies $K^A(x) \leq q(K(x))+c_0$ for some constant c_0 and all x. This needs some facts about the sequence $y_0, y_1, \ldots$ and the complexities of these strings relative to A.

For ease of notation, U^A denotes the universal prefix-free machine relative to A and $U = U^\emptyset$ the unrelativized one. Let a_n be an input of shortest length such that $U^A(a_n) = y_n$ and let b_n be an input of length y_{n-1} such that $U(b_n) = y_n$.

Now consider all the n such that $|a_n| \leq y_{n-1} - 2y_{n-2}$. Then one has a prefix-free machine V^A and a partial-recursive coding function θ such that

- $V^A(b_{n-1}a_n)$ computes $\Omega_{y_n} \upharpoonright y_{n-1} - y_{n-2} - c_1$;
- $U(\theta(b_{n-1}\Omega \upharpoonright y_{n-1} - y_{n-2} - c_1))$ computes $\min\{s : \Omega_s \upharpoonright (y_{n-1} - y_{n-2} - c_1) = \Omega \upharpoonright (y_{n-1} - y_{n-2} - c_1)\}$.

where the constant c_1 is so large that θ can be chosen such that $|\theta(b_{n-1}d)| \leq y_{n-1}$ for all $d \in \{0,1\}^{y_{n-1}-y_{n-2}-c_1}$. As a consequence, the computation $U(\theta(b_{n-1}\Omega \upharpoonright y_{n-1} - y_{n-2} - c_1))$ needs less than y_n steps. Thus, $V^A(b_{n-1}a_n)$ computes $\Omega \upharpoonright y_{n-1} - y_{n-2} - c_1$ and $|b_{n-1}a_n| = y_{n-2}+|a_n| \leq y_{n-1}-y_{n-2}$. Since Ω is random relative to A, this can happen only for finitely many n and one has that $|a_n| > y_{n-1} - 2y_{n-2}$ for almost all n.

Now assume that $n > 1$ and $|a_n| > y_{n-1} - 2y_{n-2}$. Let $E_n = \{e : U^A(e)$ needs at least $\min(I_{y_n})$ and at most $\min(I_{y_{n+1}})-1$ steps$\}$. Note that for $e \in E_n$, b_n is that string of length y_{n-1} for which $U(b_n)$ terminates last within the computation-time of $U^A(e)$ and $y_n = U(b_n)$. So one has a constant c_2 and for each e a prefix-free input d of length $|e| + K(y_{n-1}) + c_2$ such that $U^A(d) = y_n$. This gives that there is a constant c_3 with

$$\sum_{e\in E_n} 2^{-|e|-c_2-K(y_{n-1})} < 2^{c_3-|a_n|}$$

what using $|a_n| > y_{n-1} - 2y_{n_2}$ can be transformed to

$$\sum_{e\in E_n} 2^{y_{n-1}-c_2-c_3-3y_{n-2}-e} < 1.$$

There is a partial-recursive function g such that $g(b_n) = |I_{y_0} \cup I_{y_1} \cup \ldots \cup I_{y_n}|$. Now one can construct a prefix-free machine which on input bd with $U(b)$ being defined and $|d| = g(b_n)$ enumerates requests of weight at most 2^{-b-d} with the additional constraint that, in the case that $b = b_n$ and d is the restriction of A to $I_{y_0} \cup I_{y_1} \cup \ldots \cup I_{y_n}$, the requests are just an enumeration of the set

$$\{\langle |b_n| + g(b_n) + |e| + c_2 + c_3 + 3y_{n-2} - y_{n-1}, U^A(e)\rangle : e \in E_n\}.$$

Recall that the weight of a request $\langle i, j\rangle$ is 2^{-i}. So the sum of the weights of all requests is at most 1. Note from b_n and d one can compute $y_0, y_1, \ldots, y_n$ and A on $I_{y_0} \cup I_{y_1} \cup \ldots \cup I_{y_n}$ so that the enumeration is effective. By the inequality

$$\sum_{e \in E_n} 2^{y_{n-1}-c_2-c_3-3y_{n-2}-e} < 1.$$

from above one has that the bound on the weight of the requests is kept. Assume that $|e| = K^A(x)$ and $U^A(e) = x$ and x is so large that $e \in E_n$ for an n satisfying that $g(b_n) \leq 2y_{n-2}$ and that n does not fall under the finitely many exceptions considered above. Then there is a request of the form $\langle |e| + g(b_n) + c_2 + c_3 + 3y_{n-2}, x\rangle$. It follows from the Kraft-Chaitin Theorem that there is a constant c_4 with $K^A(x) \leq |e| + 5y_{n-2} + c_4$ for the n with $e \in E_n$.

As for almost all n, $|a_n| > y_{n-1} - 2y_{n-2}$ and as one can compute y_n relative to A from y_{n-2} plus an upper bound on y_n, one has that for almost all n and every e with $U^A(e)$ needing more than y_n steps that $|e| > y_{n-1} - 3y_{n-2} - c_5$ for some constant c_5. Since r grows slower than every unbounded and nondecreasing recursive function and $y_{n-1} - 3y_{n-2} - c_5 > y_{n-1}/2$ for almost all n, there is a constant c_6 such that $r(e) \geq r(y_n) - c_6 = y_{n-2} - c_6$ where c_6 is independent of e, n as long as $e \in E_n$. So one has that $K(U^A(e)) \leq |e| + 5r(|e|) + c_4 + 5c_6$.

One can now cover the case the $x = U^A(e)$ the finitely many x where $U^A(e)$ needs at most $\min(I_{y_{n+1}}) - 1$ steps for some of the finitely many exceptional n in the case distinction above by taking c_0 to be sufficiently much larger than $c_4 + 5c_6$ and obtains that

$$\forall x \; K(x) \leq K^A(x) + 5r(K^A(x)) + c_0 = q(K^A(x)) + c_0$$

what completes the proof. □

One should note that the real difficulty of this construction stems from the fact that the constructed set has to be p-low and not only weakly p-low. For estimation functions, the construction of weakly p-low sets is quite straight-forward. Note that the resulting set is not K-trivial as it is Turing complete.

Proposition 24. *Let p be an estimation function. Then there is a Turing complete r.e. set A which is weakly p-low and also satisfies the corresponding property for C: there are constants c_K, c_C such that $K(A \restriction x) \leq p(K(x)) + c_K$ and $C(A \restriction x) \leq p(C(x)) + c_C$ for all x.*

Proof. For defining an enumeration of A, fix a one-one enumeration $b_0, b_1, \ldots$ of the halting problem and approximations C_s, K_s to C, K. Let $A_0 = \emptyset$. At stage $s + 1$, let a_m be the m-th nonelement of A_s in ascending order. Now the set A_{s+1} is computed as follows.

- Let n be the minimum of all m such that one of the following conditions holds:
  - $a_m > s$;
  - $b_s \leq m$;
  - $p_s(K_s(k)) - K_s(k) \leq m$ for some k with $a_m \leq k \leq s$;
  - $p_s(C_s(k)) - C_s(k) \leq m$ for some k with $a_m \leq k \leq s$.
- Let $A_{s+1} = A_s \cup \{x : a_n \leq x \leq s\}$.

The so constructed set A satisfies the following properties:

- A is coinfinite and r.e.;
- A is Turing complete;
- $K(A \upharpoonright x) \leq p(K(x)) + c_K$ for some constant c_K and all x;
- $C(A \upharpoonright x) \leq p(C(x)) + c_C$ for some constant c_C and all x.

The first property states the obvious fact that A is r.e. by the construction. The other fact that A is co-infinite needs some more thought. Assume by way of contradiction that $|\overline{A}| = m$ for some finite number m. Let $a_0, a_1, \ldots, a_{m-1}$ denote the nonelements of A in ascending order and assume that s is so large that the following conditions hold:

- if $b_t \leq m$ then $t < s$;
- for all $x \in A - A_s$ there is no $k \geq x$ and no $e \geq \min\{C(k), K(k)\}$ such that $p(e) - e \leq m$;
- if $x \leq a_{m-1} + 1$ then $x \in A \Leftrightarrow x \in A_s$.

Then one can see that the parameters $a_0, a_1, \ldots, a_{m-1}$ chosen in the definition of step s coincide with the m least nonelements of A and are just not enumerated. Furthermore, a_m is also defined as the next nonelement of A_s. Note that $a_m \leq s$ as $s \notin A_s$. Now one can see that a_m is not enumerated into A_{s+1} because the n selected is larger than m: for all $m' < m$, $n \neq m'$ because otherwise $a_0, a_1, \ldots, a_{m-1}$ would not remain outside A; furthermore, $n \neq m$ as the first and second item in the conditions on s together with the facts that p_s approximates p from above and $a_m \leq s$ imply that m does not satisfy the search-conditions. So $a_m \notin A_{s+1}$ and one can show by induction that $a_m \notin A_t$ for all $t > s$, this contradicts the assumption that $|\overline{A}| = m$. Therefore, A is coinfinite.

The second property follows from the construction. If $a_0, a_1, \ldots$ are the nonelements of A in ascending order, then $b_s \leq m$ implies $s \leq a_m$. Thus m is in the halting problem iff $m \in \{b_0, b_1, \ldots, b_{a_m}\}$ and so the halting problem is Turing reducible to A.

The third property can be seen as follows: Given x and the shortest description σ for x with respect to a fixed prefix-free universal machine, let n be the number of nonelements of A below x. Then one can construct a prefix-free machine which from input $1^n 0 \sigma$ first evaluates the universal machine on σ to get the value x and then searches for a stage s such that A_s contains all but n elements below x. Having this x and s, the machine outputs $A_s \upharpoonright x$. If σ and n are chosen correctly, then the output is correct. Thus one has that $K(A \upharpoonright x)$ is at most $K(x)+n+c_K$ where the constant c_K comes from translating the given prefix-free coding of $K(A \upharpoonright x)$ of length $K(x)+n+1$ for some machine into inputs for the universal machine. Furthermore, for all sufficiently large s, $K_s(x)+n \leq p_s(K_s(x))$ as otherwise the marker a_{n-1} would move. Therefore $K(x)+n \leq p(K(x))$ and A is weakly p-low.

The fourth property can be proven analogously; here the constructed machine is not prefix-free and σ is the shortest input producing x with respect to some fixed universal plain machine, nevertheless σ and n can of course still be recovered from $1^n0\sigma$. The rest of the proof follows the previous item but is working with C in place of K. This completes the proof of the whole result. □

For any estimation function p and the above constructed $A \in \mathcal{K}[p]$, $\Omega \leq_T A$ and thus $A \notin \mathcal{M}[p]$ by Proposition 20 (i) and (iii). Thus the inclusion from Proposition 20 (v) is strict.

Corollary 25. *For all estimation functions p, $\mathcal{M}[p] \subset \mathcal{K}[p]$.*

Proposition 26. *For every estimation function p there is a whole Turing degree outside Δ^0_2 contained in $\mathcal{K}[p]$.*

Proof. For any estimation function p one can consider the estimation function q given as $q(n) = n + \log(p(n)-n)/2$. Then one can construct a r.e. set A as in Proposition 24 which is in $\mathcal{K}[q]$.

The set A is not recursive. Thus, due to Yates's version of the Friedberg-Muchnik Splitting Theorem [20, Theorem IX.2.4 and Exercise IX.2.5], one can construct a partial-recursive $\{0,1\}$-valued function ψ with domain A such that $\psi^{-1}(0), \psi^{-1}(1)$ form a recursively inseparable pair, that is, ψ does not have a total extension. Actually, given a one-one enumeration $a_0, a_1, \ldots$ of A, this function ψ can be inductively defined on this domain by taking $\psi(a_s)$ in $\{0,1\}$ such that $\psi(a_s)$ differs from $\varphi_{e,s}(a_s)$ for the least e where either $e = s$ or $\varphi_{e,s}(a_s)$ is defined and $\psi(a_t) = \varphi_{e,s}(a_t)$ for all $t < s$ with $a_t \in \text{domain}(\varphi_{e,s})$.

Every total extension B of ψ is in $\mathcal{K}[p]$ as given any n and any x, the number m of places below x where ψ is undefined satisfies $m < q(K(x)) - K(x)$. Let $x_1, x_2, \ldots, x_m$ be these places. Let σ be the shortest input such that the universal machine for K computes x. Then one can code $B \upharpoonright x$ by $1^m0B(x_1)B(x_2)\ldots B(x_m)\sigma$ and thus has that $K(B \upharpoonright x)$ is below $p(K(x))$. As one can take B to have hyperimmune-free Turing degree [20, Theorem V.5.34] and as $\mathcal{K}[p]$ is closed under wtt-reducibility, one has that a whole Turing degree outside Δ^0_2 is contained in $\mathcal{K}[p]$. □

Note that the above result also holds with C in place of K, the proof is exactly the same. So given an estimation function p, one can construct a hyperimmune-free Turing degree only consisting of sets E satisfying $C(E \upharpoonright x) \leq p(E(x))$ for all x up to an additive constant. Unfortunately, it is not guaranteed that this degree is also strongly jump-traceable, it is even a bit unlikely, as only the use of total E-recursive functions but not of the jump is recursively bounded in the case of a set E of hyperimmune-free degree.

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