

# Lowness Properties and Approximations of the Jump

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## Abstract

We study and compare two combinatorial lowness notions: *strong jump-traceability* and *well-approximability of the jump*, by strengthening the notion of jump-traceability and super-lowness for sets of natural numbers. A computable non-decreasing unbounded function  $h$  is called an order function. Informally, a set  $A$  is strongly jump-traceable if for each order function  $h$ , for each input  $e$  one may effectively enumerate a set  $T_e$  of possible values for the jump  $J^A(e)$ , and the number of values enumerated is at most  $h(e)$ .  $A'$  is well-approximable if can be effectively approximated with less than  $h(x)$  changes at input  $x$ , for each order function  $h$ . We prove that there is a strongly jump-traceable set which is not computable, and that if  $A'$  is well-approximable then  $A$  is strongly jump-traceable. For r.e. sets, the converse holds as well. We characterize jump-traceability and strong jump-traceability in terms of Kolmogorov complexity. We also investigate other properties of these lowness properties.

## 1 Introduction

A *lowness property* of a set  $A$  says that  $A$  is computational weak when used as an oracle and hence  $A$  is close to being computable. In this article we study and compare some “combinatorial” lowness properties in the direction of characterizing  $K$ -trivial sets.

A set is  $K$ -trivial when it is highly compressible in terms of Kolmogorov complexity (see Section 2 for the formal definition). In [19], Nies proved that a set is  $K$ -trivial if and only if  $A$  is low for Martin-Löf-random (that is, each Martin-Löf-random set is already random relative to  $A$ ).

Terwijn and Zambella [24] defined a set  $A$  to be *recursively traceable* if there is a recursive bound  $p$  such that for every  $f \leq_T A$ , there is a recursive  $r$  such that for all  $x$ ,  $|D_{r(x)}| \leq p(x)$ , and  $(D_{r(x)})_{x \in \mathbb{N}}$  is a set of possible values of  $f$ : for all  $x$ , we have  $f(x) \in D_{r(x)}$ .

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They showed that this combinatorial notion characterizes the sets that are low for Schnorr tests.

This property was modified in [20] to *jump-traceability*. A set  $A$  is jump traceable if its jump at argument  $e$ , written  $J^A(e) = \{e\}^A(e)$ , has few possible values.

**Definition 1.** A uniformly r.e. family  $T = \{T_0, T_1, \dots\}$  of sets of natural numbers is a *trace* if there is a recursive function  $h$  such that  $\forall n |T_n| \leq h(n)$ . We say that  $h$  is a *bound* for  $T$ . The set  $A$  is *jump-traceable* if there is a trace  $T$  such that

$$\forall e [J^A(e) \downarrow \Rightarrow J^A(e) \in T_e].$$

We say that  $A$  is jump traceable *via a function*  $h$  if, additionally,  $T$  has bound  $h$ .

Another notion studied in [20] is *super-lowness*, first introduced in [4, 18].

**Definition 2.** A set  $A$  is  $\omega$ -r.e. iff there exists a recursive function  $b$  such that  $A(x) = \lim_{s \rightarrow \infty} g(x, s)$  for a recursive  $\{0, 1\}$ -valued  $g$  such that  $g(x, s)$  changes at most  $b(x)$  times, that is,  $|\{s : g(x, s) \neq g(x, s + 1)\}| \leq b(x)$ . In this case, we say that  $A$  is  $\omega$ -r.e. *via the function*  $g$  and *bound*  $b$ .  $A$  is *super-low* iff  $A'$  is an  $\omega$ -r.e. set.

Recall that a set  $A$  is *low* if  $A' \leq_T \emptyset'$ . The above definition of  $A$  being super-low is equivalent to  $A' \leq_{tt} \emptyset'$ . Hence super-lowness implies lowness.

Both the classes of jump-traceable and of super-low sets are closed downward under Turing reducibility and contained in the class of generalized low sets  $\{A : A' \leq A \oplus \emptyset'\}$ . In [20] it was proven that these two lowness notions coincide within the r.e. sets but that none of them implies the other within the  $\omega$ -r.e. sets.

In this article, we define the notions of *strong jump-traceability* and *well-approximability of the jump*, strengthening super-lowness. In the strong variant of these notions we consider *all* order functions as the bound instead of just *some* recursive bound. Here, an *order* function is a recursive, non-decreasing and unbounded function (intuitively, think of a *slowly growing* but unbounded recursive function). Our first two results are:

- There is a non-computable strongly jump-traceable set;
- If  $A'$  is well-approximable then  $A$  is strongly jump-traceable; the converse also holds, if  $A$  is an r.e. set.

Our approach is used to study interesting lowness properties related to plain and prefix-free Kolmogorov complexity. We investigate the properties of sets  $A$  such that Kolmogorov complexity relative to  $A$  is only a bit smaller than the unrelativized one. We prove some characterizations of jump-traceability and strong jump-traceability in terms of prefix-free (denoted by  $K$ ) and plain (denoted by  $C$ ) Kolmogorov complexity, respectively:

- $A$  is jump-traceable if and only if there is a recursive  $p$ , growing faster than linearly such that  $K(y)$  is bounded by  $p(K^A(y) + c_0) + c_1$ , for some constants  $c_0$  and  $c_1$ ;

- $A$  is strongly jump-traceable if and only if  $C(x) - C^A(x)$  is bounded by  $h(C^A(x))$ , for every order function  $h$  and almost all  $x$ .

Recall that  $A$  is low for  $K$  iff  $K(x) \leq K^A(x) + O(1)$  for each  $x$ . Nies [19] has shown that this property is equivalent to being  $K$ -trivial. In particular, non-computable low for  $K$  sets exist. The corresponding property involving  $C$  is only satisfied by the computable sets (because it implies being  $C$ -trivial by [7], which is the same as computable). The characterization of strongly jump-traceable is via a property that states that  $C^A$  is very close to  $C$ , while not implying computability.

By [19],  $K$ -triviality implies jump-traceability. Recently, Cholak, Downey and Greenberg [9] have shown that for r.e. sets  $A$ , strong jump-traceability implies  $K$ -triviality. They also prove that there is a  $K$ -trivial r.e. set that is not strongly jump-traceable.

## 2 Basic definitions

If  $A$  is a set of natural numbers then  $A(x) = 1$  if  $x \in A$ ; otherwise  $A(x) = 0$ . We denote by  $A \upharpoonright n$  the string of length  $n$  which consists of the bits  $A(0) \dots A(n-1)$ .

If  $A$  is given by an effective approximation and  $\Psi$  is a functional, we write  $\Psi^A(e)[s]$  for  $\Psi_s^{A_s}(e)$ . From a partial recursive functional  $\Psi$ , one can effectively obtain a primitive recursive and strictly increasing function  $\alpha$ , called a *reduction function* for  $\Psi$ , such that

$$\forall X \forall e \Psi^X(e) = J^X(\alpha(e)).$$

For each set  $A$ , we want to define  $K^A(y)$  as the length of a shortest prefix-free description of  $y$  using oracle  $A$ . An *oracle machine* is a partial recursive functional  $M : \{0, 1\}^\infty \times \{0, 1\}^* \mapsto \{0, 1\}^*$ . We write  $M^A(x)$  for  $M(A, x)$ .  $M$  is an *oracle prefix-free machine* if the domain of  $M^A$  is an antichain under inclusion of strings, for each  $A$ . Let  $(M_d)_{d \in \mathbb{N}}$  be an effective listing of all oracle prefix-free machines. The universal oracle prefix-free machine  $U$  is given by

$$U^A(0^d 1 \sigma) = M_d^A(\sigma)$$

and the prefix-free Kolmogorov complexity relative to  $A$  is defined as

$$K^A(y) = \min\{|\sigma| : U^A(\sigma) = y\},$$

where  $|\sigma|$  denotes the length of  $\sigma$ . If  $A = \emptyset$ , we simply write  $U(\sigma)$  and  $K(y)$ . As usual,  $U(\sigma)[s] \downarrow = y$  indicates that  $U(\sigma) = y$  and the computation takes at most  $s$  steps. Schnorr's Theorem states that  $A \in \{0, 1\}^\infty$  is Martin-Löf random iff the initial segments of  $A$  have high  $K$ -complexity, that is,

$$\exists c \forall n K(A \upharpoonright n) > n - c.$$

A set  $A$  is  $K$ -trivial iff the initial segments of  $A$  have low  $K$ -complexity, that is,

$$\exists c \forall n K(A \upharpoonright n) \leq K(n) + c.$$

We say that  $A \leq_K B$  iff

$$\exists c \forall n K(A \upharpoonright n) \leq K(B \upharpoonright n) + c.$$

The Kraft-Chaitin Theorem states that from a recursive sequence of pairs  $(\langle n_i, \sigma_i \rangle)_{i \in \mathbb{N}}$  (known as *requests*) such that  $\sum_{i \in \mathbb{N}} 2^{-n_i} \leq 1$ , we can effectively obtain a prefix-free machine  $M$  such that for each  $i$  there is a  $\tau_i$  of length  $n_i$  with  $M(\tau_i) \downarrow = \sigma_i$ , and  $M(\rho) \uparrow$  unless  $\rho = \tau_i$  for some  $i$ .

If we drop the condition of the domain of  $M^A$  being an antichain, we obtain a similar notion, called plain Kolmogorov complexity denoted by  $C$ . Hence,  $C^A(y)$  will denote the length of the shortest description of  $y$  using oracle  $A$ , when we do not have the restriction on the domain.

A *binary machine* is a partial recursive function  $\tilde{M} : \{0, 1\}^* \times \{0, 1\}^* \mapsto \{0, 1\}^*$ . Let  $\tilde{U}$  be a binary universal function given by

$$\tilde{U}(0^d 1 \sigma, x) = \tilde{M}_d(\sigma, x),$$

where  $(\tilde{M}_d)_{d \in \mathbb{N}}$  is an enumeration of all partial recursive functions of two arguments. We define the plain conditional Kolmogorov complexity  $C(y|x)$  as the length of the shortest description of  $y$  using  $\tilde{U}$  with string  $x$  as the second argument, that is,

$$C(y|x) = \min\{|\sigma| : \tilde{U}(\sigma, x) = y\}.$$

Let  $str : \mathbb{N} \rightarrow \{0, 1\}^*$  be the standard enumeration of the strings. The string  $str(n)$  is that binary sequence  $b_0 b_1 \dots b_m$  for which the binary number  $1b_0 b_1 \dots b_m$  has the value  $n + 1$ . Thus,  $str(0) = \lambda$ ,  $str(1) = 0$ ,  $str(2) = 1$ ,  $str(3) = 00$ ,  $str(4) = 01$  and so on.

### 3 Strong jump-traceability

Recall that an r.e. set  $A$  is *promptly simple* if  $A$  is co-infinite and there is a recursive function  $p$  and an effective approximation  $(A_s)_{s \in \mathbb{N}}$  of  $A$  such that, for each  $e$ ,

$$|W_e| = \infty \Rightarrow \exists s \exists x [x \in W_{e,s+1} \setminus W_{e,s} \wedge x \in A_{p(s)}]. \quad (1)$$

In this section, we introduce a stronger version of jump-traceability and we prove that there is a promptly simple (hence non-recursive) strongly jump-traceable set. We also prove that there is no single maximal order function that suffices as the bounding function for all instances of jump-traceability.

**Definition 3.** A computable function  $h : \mathbb{N} \rightarrow \mathbb{N}^+$  is an *order function* if  $h$  is non-decreasing and unbounded.

Notice that any reduction function is an order function.

**Definition 4.** A set  $A$  is *strongly jump-traceable* iff for each order function  $h$ ,  $A$  is jump traceable via  $h$ .

Clearly, strong jump-traceability implies jump-traceability. It is not difficult to see that strong jump-traceability is closed downward under Turing reducibility.

**Proposition 5.**  $\{A : A \text{ is strongly jump-traceable}\}$  is closed downward under Turing reducibility.

**Proof.** Suppose  $A$  is strongly jump-traceable,  $B \leq_T A$ . We prove that  $B$  is jump-traceable via the given order function  $h$ . Let  $\Psi$  be the functional such that  $\Psi^A(x) = J^B(x)$  for all  $x$  and let  $\alpha$  be the reduction function such that  $J^A(\alpha(x)) = \Psi^A(x)$ . We know that  $A$  is jump-traceable via a trace  $(T_i)_{i \in \mathbb{N}}$  with bound  $\tilde{h}$ , where  $\tilde{h}(z) = h(\min\{y : y \in \mathbb{N} \wedge \alpha(y+1) \geq z\})$ . Observe that, since  $\alpha$  is an order function,  $\tilde{h}$  also is. Clearly,

$$J^B(e) = J^A(\alpha(e)) \downarrow \Rightarrow J^B(e) \in T_{\alpha(e)}.$$

Now,  $\tilde{h}(\alpha(e)) = h(y)$  for some  $y$  such that  $\alpha(y) < \alpha(e)$  or  $y = 0$ . Then  $y \leq e$  and  $\tilde{h}(\alpha(e)) = h(y) \leq h(e)$ . Hence  $(T_{\alpha(i)})_{i \in \mathbb{N}}$  is a trace for the jump of  $B$  with bound  $h$ .  $\square$

Clearly each computable set  $A$  is strongly jump-traceable, because we can trace the jump by

$$T_e = \begin{cases} \{J^A(e)\} & \text{if } J^A(e) \downarrow; \\ \emptyset & \text{otherwise.} \end{cases}$$

In Theorem 7 below we show the existence of a non-computable strongly jump-traceable set. We need the following result, proven in [17, Theorem 2.3.1]:

**Lemma 6.** *The function  $m(x) = \min\{C(y) : y \geq x\}$  is unbounded, non-decreasing and for every order function  $f$  there is an  $x_0$  such that  $m(x) < f(x)$  for all  $x \geq x_0$ . Also,  $m(x) = \lim_{s \rightarrow \infty} m_s(x)$ , where  $m_s(x) = \min\{C_s(y) : x \leq y \leq x + s\}$  is recursive and  $m_s(x) \geq m_{s+1}(x)$ , for all  $x$  and  $s$ .*

Observe that here  $\lambda x, s. C_s(x)$  is the standard recursive approximation from above of  $C(x)$  (that is  $\lambda s. C_s(x) \rightarrow C(x)$  when  $s \rightarrow \infty$  and  $C_s(x) \geq C_{s+1}(x)$ ).

**Theorem 7.** *There exist a promptly simple strongly jump-traceable set.*

**Proof.** We construct a promptly simple set  $A$  in stages satisfying the requirements

$$P_e : |W_e| = \infty \Rightarrow \exists s \exists x [x \in W_{e,s+1} \setminus W_{e,s} \wedge x \in A_{s+1}].$$

These requirements will ensure that  $A$  is promptly simple (indeed, take  $p(s) = s + 1$  in equation (1)). Each time we enumerate an element into  $A$  in order to satisfy  $P_e$ , we may destroy  $J^A(k)$  and then our trace for the jump of  $A$  will grow. Hence, we must enumerate elements into  $A$  in a controlled way, and sometimes we should refrain from putting elements into  $A$ . Since for any order function  $h$  there has to be a trace for  $J^A$  bounded by  $h$ , we will work with the function  $m$  defined in Lemma 6, which grows slower than any order function. The rule will be that during the construction,  $P_e$  may destroy  $J^A(k)$  at stage  $s$

only if  $e < m_s(k)$ . (Observe that the restriction on  $P_e$  imposed rule may strengthen as  $s$  grows, because we may have  $m_s(k) > m_{s+1}(k)$ .) In this way, we will guarantee that size of our trace for  $J^A(e)$  will be bounded by  $m(e)$ , which will suffice because  $m \leq h$  from some point on. As we will see, the exact choice of the trace for  $J^A$  with bound  $h$  depends on  $h$ , and is made in a nonuniform way.

In the following construction we use the convention that  $W_{e,s} \subseteq \{0, 1, \dots, s\}$  for all indices  $e$  and stages  $s$ .

**Construction of A.** Let  $m_s$  be the non-decreasing, unbounded function defined in Lemma 6.

*Stage 0:* set  $A_0 = \emptyset$  and declare  $P_e$  unsatisfied for all  $e$ .

*Stage  $s + 1$ :* choose the least  $e \leq s$  such that

- $P_e$  yet not satisfied;
- There exists  $x$  such that  $x \in W_{e,s+1} \setminus W_{e,s}$ ,  $x > 2e$  and for all  $k$  such that  $m_s(k) \leq e$ , if  $J^A(k)[s]$  is defined then  $x$  is greater than the use of  $J^A(k)[s]$ .

If such  $e$  exists, put the least such  $x$  into  $A$  for each such  $e$ . We say that  $P_e$  receives attention at stage  $s + 1$  and declare  $P_e$  satisfied. Otherwise,  $A_{s+1} = A_s$ . Finally, define  $A = \bigcup_s A_s$ .

**Verification.** Clearly,  $P_e$  receives attention at most once. So we can use below the fact that every requirement influences the enumeration of  $A$  at most once.

To show that  $A$  is strongly jump-traceable, fix a recursive order function  $h$ . We will prove that there exists an r.e. trace  $T$  for  $J^A$  as in Definition 1. Let  $h$  be any order function. By Lemma 6, there exists  $k_0$  such that for all  $k \geq k_0$ ,  $m(k) \leq h(k)$ . Define the recursive function

$$f(k) = \begin{cases} \min\{s : m_s(k) \leq h(k)\} & \text{if } k \geq k_0; \\ 0 & \text{otherwise.} \end{cases}$$

For  $k \geq k_0$  and  $s \geq f(k)$ ,  $m_s(k)$  will be below  $h(k)$ , so  $J^A(k)$  may change because  $P_e$  receives attention, for  $e < m_s(k) \leq h(k)$ . Since each  $P_e$  receives attention at most once,  $J^A(k)$  can change at most  $h(k)$  times after stage  $f(k)$ . So

$$T_k = \begin{cases} \{J^A(k)[s] : J^A(k)[s] \downarrow \wedge s \geq f(k)\} & \text{if } k \geq k_0; \\ \{J^A(k)\} & \text{if } J^A(k) \downarrow \wedge k < k_0; \\ \emptyset & \text{otherwise.} \end{cases}$$

is as required.

Fix  $e$  such that  $W_e$  is infinite and let us see that  $P_e$  is met. Let  $s$  such that

$$\forall k [m(k) \leq e \Rightarrow m_s(k) = m(k)]$$

and  $s' > s$  such that no  $P_i$  receives attention after stage  $s'$  for any  $i < e$ . Then, by the construction, no computation  $J^A(k)$ ,  $m(k) \leq e$  can be destroyed after stage  $s'$ . So there

is  $t > s'$  such that for all  $k$  where  $m_t(k) \leq e$ , if  $J^A(k)$  converges then the computation is stable from stage  $t$  on. Choose  $t' \geq t$  such that there is  $x \in W_{e,t'+1} \setminus W_{e,t'}$ ,  $x > 2e$  and  $x$  is greater than the use of all converging  $J^A(k)$  for all  $k$  where  $m_{t'}(k) \leq e$ . Now either  $P_e$  was already satisfied or  $P_e$  receives attention at stage  $t' + 1$ . In either case  $P_e$  is met.  $\square$

Next we study the size of the trace bound for jump-traceability. Given an order function  $h$ , it is always possible to find a jump-traceable set  $A$  for which  $h$  is too small to be a bound for any trace for the jump of  $A$ .

**Theorem 8.** *For any order function  $h$  there is an r.e. set  $A$  and an order function  $\tilde{h}$  such that  $A$  is jump-traceable via  $\tilde{h}$  but not via  $h$ .*

**Proof.** We will define an auxiliary functional  $\Psi$  and we use  $\alpha$ , the reduction function for  $\Psi$  (that is,  $\Psi^X(e) = J^X(\alpha(e))$  for all  $X$  and  $e$ ), in advance by the Recursion Theorem. At the same time, we will define an r.e. set  $A$  and a trace  $\tilde{T}$  for  $J^A$ . Finally, we will verify that there is an order function  $\tilde{h}$  as stated.

Let  $T(0), T(1), \dots$  be an enumeration of all the traces with bound  $h$ , so that

$$T(e) = \{T(e)_0, T(e)_1, \dots\},$$

the  $e$ -th such trace, is as in Definition 1. Requirement  $P_e$  tries to show that  $J^A$  is not traceable via the trace  $T(e)$  with bound  $h$ , that is,

$$P_e : \exists x \Psi^A(x) \notin T(e)_{\alpha(x)}$$

and requirement  $N_e$  tries to stabilize the jump when it becomes defined, that is,

$$N_e : [\exists^\infty s J^A(e)[s] \downarrow] \Rightarrow J^A(e) \downarrow.$$

The strategy for a single procedure  $P_e$  consists of an initial action and a possible later action.

**Initial action at stage  $s + 1$ :**

- Choose a new candidate  $x_e = \langle e, n \rangle$ , where  $n$  is the number of times that  $P_e$  has been initialized. Define  $\Psi^A(x_e)[s + 1] = 0$  with large use.

**Action at stage  $s + 1$ :**

- Let  $x_e = \langle e, n \rangle$  be the current candidate. Put  $y$  into  $A_{s+1}$ , where  $y$  is the use of the defined  $\Psi^A(x_e)[s]$ . Notice that in the construction this action will not affect  $J^A(i)[s]$  for  $i < e$  because of the choice of  $y$ ;
- Define  $\Psi^A(x_e)[s + 1] = \Psi^A(x_e)[s] + 1$  with use  $y' > y$  and greater than the use of all defined computations of  $J^A(i)[s + 1]$  for  $i < e$ .

We say that  $P_e$  *requires attention* at stage  $s + 1$  if  $\Psi^A(x_e)[s] \in T(e)_{\alpha(x_e)}[s]$  and we say that  $N_e$  *requires attention* at stage  $s + 1$  if  $J^A(e)[s]$  becomes defined for the first time.

**Construction of A.** We define  $\tilde{T} = \{\tilde{T}_0, \tilde{T}_1, \dots\}$  by stages. The  $s$ -th stage of  $\tilde{T}_i$  will be denoted by  $\tilde{T}_i[s]$ . We start with  $A_0 = \emptyset$  and  $\tilde{T}_i[0] = \emptyset$  for all  $i$ . At stage  $s + 1$  we consider the procedures  $N_j$  for  $j \leq s$  and  $P_j$  for  $j < s$ . We also initialize the new  $P_s$ . We look at the least procedure requiring attention in the order

$$P_0, N_0, \dots, P_s, N_s.$$

If there is none, do nothing. Otherwise, suppose  $P_e$  is the first one. We let  $P_e$  take action at  $s + 1$ , changing  $A$  below the use of  $\Psi^A(x_e)[s]$  and redefining  $\Psi^A(x_e)[s + 1]$  without affecting  $N_i$  for  $i < e$ . We keep the other computations of  $P_j$  with the new definition of  $A$ , for  $j \neq i$  and large use. If  $N_e$  is the least procedure requiring attention, there is  $y$  such that  $J^A(e)[s] \downarrow = y$ . We put  $y$  into  $\tilde{T}_e[s + 1]$  and initialize  $P_j$  for  $e < j \leq s$ . In this case, we say that  $N_e$  *acts*.

**Verification.** Let us prove that  $P_e$  is met. Take  $s$  such that all  $J^A(i)$  are stable for  $i < e$ . Suppose  $x_e$  is the actual candidate of  $P_e$ . Since  $P_e$  is not going to be initialized again,  $x_e$  is the last candidate it picks. Each time  $\Psi^A(x_e)[t] \in T(e)_{\alpha(x_e)}[t]$  for  $t > s$ ,  $P_e$  acts and changes the definition of  $\Psi^A(x_e)$  to escape from  $T(e)_{\alpha(x_e)}$ . Since  $|T(e)_{\alpha(x_e)}| \leq h(\alpha(x_e))$ , there is  $s' > s$  such that  $T(e)_{\alpha(x_e)}[s'] = T(e)_{\alpha(x_e)}$ . By construction,  $\Psi^A(x_e)[s' + 1] \notin T(e)_{\alpha(x_e)}$  and  $\Psi^A(x_e)[s' + 1]$  is stable.

We say that  $N_e$  is *injured* at stage  $s + 1$  if we put  $y$  into  $A_{s+1}$  and  $y$  is less or equal than the use of  $J^A(e)[s]$ . We define  $c_P(k)$  as a bound for the number of initializations of  $P_r$ , for  $r \leq k$ ; and define  $c_N(k)$  as a bound for the number of injuries to  $N_r$ , for  $r \leq k$ . Since  $P_0$  is initialized just once and makes at most  $h(\langle 0, 0 \rangle)$  changes in  $A$ ,  $c_P(0) = 1$  and  $c_N(0) = h(\langle 0, 0 \rangle)$ . The number of times that  $P_{k+1}$  is initialized is bounded by the number of times that  $N_r$  acts, for  $r \leq k$ , so

$$c_P(k + 1) = c_P(k) + c_N(k).$$

Each time  $N_r$  is injured, for  $r \leq k$  then  $N_{k+1}$  may also be injured; additionally,  $N_{k+1}$  may be injured each time  $P_{k+1}$  changes  $A$ . The latter occurs at most  $h(\langle k + 1, i \rangle)$  for the  $i$ -th initialization of  $P_{k+1}$ . Hence

$$c_N(k + 1) = 2c_N(k) + \sum_{i \leq c_P(k+1)} h(\langle k + 1, i \rangle).$$

Once  $N_e$  is not injured anymore, if  $J^A(e) \downarrow$  then  $J^A(e) \in \tilde{T}_e$ . Since the number of changes of  $J^A(k)$  is at most the number of injuries to  $N_e$ , we define the function  $\tilde{h}(e) = c_N(e)$  which is clearly an order function and it constitutes a bound for the trace  $(\tilde{T}_i)_{i \in \mathbb{N}}$ .  $\square$

It is open whether there is *minimal* bound for jump-traceability. That is, given an order function  $h$ , is there a set  $A$  and an order function  $\tilde{h}$  such that  $A$  is jump-traceable via  $h$  but not via a smaller function  $\tilde{h}$ ? If the answer is negative for some order function  $h$ , then strong jump traceability is equivalent jump traceability for that single function  $h$ .

## 4 Well-approximability of the jump

We strengthen the notion of super-lowness and study the relationship to strongly jump-traceability.

**Definition 9.** A set  $D$  is *well-approximable* iff for each order function  $b$ ,  $D$  is  $\omega$ -r.e. via  $b$ .

Clearly, if  $A'$  is well-approximable, then  $A$  is super low. It is not difficult to see that well-approximability of the jump is closed downward under Turing reducibility.

**Proposition 10.**  $\{A : A' \text{ is well approximable}\}$  is closed downward under Turing reducibility.

**Proof.** Suppose  $A$  is such that  $A'$  is well-approximable and let  $B \leq_T A$ . We prove that  $B'$  is well-approximable via the given order function  $b$ . Define  $\Psi$  and  $\alpha$  as in Proposition 5. We know that there is a recursive  $\{0, 1\}$ -valued  $g$  such that  $A'(x) = \lim_{s \rightarrow \infty} g(x, s)$  and  $g(x, s)$  changes at most  $\tilde{b}(x)$  times, where  $\tilde{b}(z) = b(\min\{y : y \in \mathbb{N} \wedge \alpha(y+1) \geq z\})$ . Then

$$\lim_{s \rightarrow \infty} g(\alpha(x), s) = A'(\alpha(x)) = B'(x)$$

and  $g(\alpha(x), s)$  changes at most  $\tilde{b}(\alpha(x))$  times. As in Proposition 5,  $\tilde{b}(\alpha(x)) \leq b(x)$ .  $\square$

We next prove that if  $A$  is r.e., then  $A$  is strongly jump-traceable iff  $A'$  is well-approximable. We will need the following two lemmas.

**Lemma 11.** Let  $f$  and  $\hat{f}$  be order functions such that  $f(x) \leq \hat{f}(x)$  for almost all  $x$ .

- (i) If  $A$  is jump-traceable via  $f$  then  $A$  is jump traceable via  $\hat{f}$ ;
- (ii) If  $A$  is well-approximable via  $f$  then  $A$  is well-approximable via  $\hat{f}$ .

**Proof.** Assume that  $\exists x_0 \forall x [x \geq x_0 \Rightarrow f(x) \leq \hat{f}(x)]$ . For (i), suppose  $T$  is a trace for  $J^A$  with bound  $f$ . We can define the trace  $\hat{T}$ :

$$\hat{T}_x = \begin{cases} T_x & \text{if } x \geq x_0; \\ \{J^A(x)\} & \text{otherwise} \end{cases}$$

Hence, if  $x \geq x_0$  then  $|\hat{T}_x| = |T_x| \leq f(x) \leq \hat{f}(x)$ , and if  $x < x_0$  then  $1 = |\hat{T}_x| \leq \hat{f}(x)$ .

For (ii), suppose  $A$  is well-approximable via the  $\{0, 1\}$ -valued  $g(x, s)$  which changes at most  $f(x)$  times. Define

$$\hat{g}(x, s) = \begin{cases} g(x, s) & \text{if } x \geq x_0; \\ A(x) & \text{otherwise.} \end{cases}$$

If  $x \geq x_0$  then  $\hat{g}(x, s)$  changes at most  $f(x) \leq \hat{f}(x)$  times, and if  $x < x_0$  then  $\hat{g}$  does not change at all.  $\square$

**Lemma 12.** *There exists a recursive  $\gamma$  such that for all r.e.  $A$ :*

- (i) *If  $A$  is jump-traceable via an order function  $h$  then  $A$  is super-low via the order function  $b(x) = 2h(\gamma(x)) + 2$ ;*
- (ii) *If  $A$  is super-low via an order function  $b$  then  $A$  is jump-traceable via the order function  $h(x) = \lfloor \frac{1}{2}b(\gamma(x)) \rfloor$ .*

**Proof.** We follow the proof of [20, Theorem 4.1], together with Lemma 11.

(i) $\Rightarrow$ (ii). Suppose  $A$  is jump-traceable via  $h$ . By [20]  $A$  is super-low via a  $\{0, 1\}$ -valued recursive  $g$  such that  $g(x, s)$  changes at most  $2h(\alpha(x)) + 2$  times. Here,  $\alpha$  is a reduction function (hence primitive recursive) which depends on  $A$ . The diagonal  $\gamma$  of the Ackermann-function satisfies  $\gamma(x) \geq \alpha(x)$  for almost all  $x$  [21, Volume 2, Theorem VIII.8.10]. Since  $h$  is an order function,  $2(h \circ \gamma) + 2$  also is, and  $2h(\gamma(x)) + 2 \geq 2h(\alpha(x)) + 2$  for almost all  $x$ . By Lemma 11,  $A$  is super-low via  $b(x) = 2h(\gamma(x)) + 2$ .

(ii) $\Rightarrow$ (i). Suppose  $A$  is super-low via an order function  $b$  and the  $\{0, 1\}$ -valued function  $g$ . Again following [20], there is a trace for  $J^A$  via  $\lfloor \frac{1}{2}(b \circ \gamma) \rfloor$ , for a primitive recursive  $\alpha$  which depends on  $g$ . As we did in the previous implication,  $\lfloor \frac{1}{2}b(\gamma(x)) \rfloor \geq \lfloor \frac{1}{2}b(\alpha(x)) \rfloor$  for almost all  $x$ . Thus  $A$  is jump-traceable via  $h(x) = \lfloor \frac{1}{2}b(\gamma(x)) \rfloor$ .  $\square$

**Theorem 13.** *Let  $A$  be an r.e. set. Then the following are equivalent:*

- (i)  *$A$  is strongly jump-traceable;*
- (ii)  *$A'$  is well-approximable.*

**Proof.** (i) $\Rightarrow$ (ii). Given an order function  $b$ , let us prove that  $A$  is super-low via  $b$ . By part (i) of Lemma 12, it suffices to define an order function  $h$  such that  $2h(\gamma(x)) + 2 \leq b(x)$  for almost all  $x$ . If  $b(x) \geq 4$  then define  $h(\gamma(x)) = \lfloor \frac{b(x)-2}{2} \rfloor$  and if  $b(x) < 4$ , define  $h(\gamma(x)) = 1$ . Since  $\gamma$  can be taken strictly monotone, the above definition is correct and we can complete it to make  $h$  an order function.

(ii) $\Rightarrow$ (i). Given an order function  $h$ , we will prove that  $A$  is jump-traceable via  $h$ . By part (ii) of Lemma 12, it suffices to define an order function  $b$  such that  $\lfloor \frac{1}{2}b(\gamma(x)) \rfloor \leq h(x)$  for almost all  $x$ . The argument is similar to the previous case.  $\square$

Later, in Corollary 18, we will improve this result and we will see that, in fact, the implication (ii) $\Rightarrow$ (i) holds for any  $A$ .

We finish this section by proving that the prefixes  $D \upharpoonright n$  of a well-approximable set  $D$  have low Kolmogorov complexity, of order logarithmic in  $n$ . Hence  $D$  is not Martin-Löf random and furthermore, its effective Hausdorff dimension is 0. The latter is equivalent to saying that there is no  $c > 0$  such that  $cn$  is a linear lower bound for the prefix-free Kolmogorov complexity of  $D \upharpoonright n$  for almost all  $n$ . In the following  $|n|$  denotes the length of the binary representation of  $n$ .

**Theorem 14.** *If  $D$  is well-approximable then for almost all  $n$ ,  $K(D \upharpoonright n) \leq 4|n|$ .*

**Proof.** Suppose  $D(n) = \lim_{s \rightarrow \infty} g(n, s)$ , where  $g$  is recursive and changes at most  $n$  times. Given  $n$ , there is a unique  $s$  and some  $m < n$  such that  $g(m, s) \neq g(m, s + 1)$  but  $g(q, t) = g(q, t + 1)$  for all  $t > s$  and  $q < n$ . That is,  $s$  is the first stage where  $g(0, s + 1) = D(0), \dots, g(n - 1, s + 1) = D(n - 1)$  and  $m$  is the place where the last change takes place. The stage  $s$  can be computed from  $m$  and the number  $k$  of stages with  $g(m, t) \neq g(m, t + 1)$ . So one can compute  $D \upharpoonright n$  from  $m, n, k$ . Since  $k, m \leq n$ , one can, for almost all  $n$ , code  $m, n, k$  in a prefix-free way in  $4|n|$  many bits. This is done by using a prefix of the form  $1^q 0$  followed by  $2q$  bits representing  $n$ ,  $2q$  bits representing  $m$  and  $2q$  bits representing  $k$  as binary numbers; here  $q$  is just the smallest number such that  $2q$  bits are enough. Since  $k, m \leq n$  and since  $2q \leq |n| + c$  for some constant  $c$  and since the additionally necessary coding needed to transform the above representation into a program for  $U$  is bounded by a constant, we have that there is a constant  $d$  such that

$$\forall n \ K(D \upharpoonright n) \leq 3|n| + |n|/2 + d$$

and then the relation  $K(D \upharpoonright n) \leq 4|n|$  holds for almost all  $n$ . In fact, using binary notation to store  $q$  instead of  $1^q 0$ , it would even give

$$K(D \upharpoonright n) \leq 3(|n| + \log(|n|))$$

for almost all  $n$ . □

## 5 Traceability and plain Kolmogorov complexity

We give a characterization of strong jump-traceability in terms of (relativized) plain Kolmogorov complexity. First we show that if  $A'$  is well-approximable then  $A$  satisfies the condition involving Kolmogorov complexity and hence that any set  $A$  such that  $A'$  is well-approximable is strongly jump-traceable.

**Theorem 15.** *If  $A'$  is well-approximable then for every order function  $h$  and almost all  $x$ ,  $C(x) \leq C^A(x) + h(C^A(x))$ .*

**Proof.** The idea of the proof is the following. Let  $h$  be any order function. Suppose  $q_x$  is a minimal  $A$ -program for  $x$ . We know that there is a  $c$  such that  $C(x) \leq |q_x| + 2C(x|q_x) + c$ . Since  $|q_x| = C^A(x)$ , we only need to show that  $2C(x|q_x) + c \leq h(|q_x|)$  for almost all  $x$ . Given  $q_x$  and the value of  $C(x|q_x)$ , we can find a program  $p_x$  of length  $C(x|q_x)$  which describes  $x$  with the help of  $q_x$ , that is  $\tilde{U}(p_x, q_x) = x$ . It can be shown that there is a recursive  $\{0, 1\}$ -valued approximation of the bits of  $p_x$  which changes few times (in the proof, this is done with the help of the functional  $\Psi$ ). Hence,  $x$  can be described by the values of  $C(x|q_x)$ ,  $q_x$  and  $p_x$ . We can represent  $p_x$  with the number of changes of the mentioned  $\{0, 1\}$ -valued approximation. This will show  $C(x|q_x) \leq 2|h(|q_x|)| + O(1)$ , which is sufficient to get the desired upper bound on  $2C(x|q_x) + c$ .

Here are the details. Let  $\Psi^A(m, n, q)$  be a functional which does the following:

- (i) Compute  $x = U^A(q)$ . If  $U^A(q) \uparrow$  then  $\Psi^A(m, n, q) \uparrow$ ;

- (ii) Find the first program  $p$  such that  $|p| = n$  and  $\tilde{U}(p, q) = x$ . If there is no such  $p$  then  $\Psi^A(m, n, q) \uparrow$ ;
- (iii) In case  $m \notin [1, n]$  then  $\Psi^A(m, n, q) \uparrow$ . Otherwise, if the  $m$ -th bit of  $p$  is 1 then  $\Psi^A(m, n, q) \downarrow$ , else  $\Psi^A(m, n, q) \uparrow$ .

Let  $\alpha$  be a reduction function such that  $J^A(\alpha(m, n, q)) = \Psi^A(m, n, q)$ . Choose an order function  $b$  such that  $b(\alpha(n, n, q)) \leq nh(|q|)$  for all  $n, q$ . We can approximate  $A'(x)$  with a  $\{0, 1\}$ -valued recursive function which changes at most  $b(x)$  times.

Let  $q_x$  be a minimal  $A$ -program for  $x$ , that is,  $U^A(q_x) = x$  and  $|q_x| = C^A(x)$ . Let  $n_x = C(x|q_x)$ . Then  $\Psi^A(m, n_x, q_x) \downarrow$  iff the  $m$ -th bit of  $p_x$  is 1, where  $p_x$  is the first program such that  $|p_x| = n_x$  and  $\tilde{U}(p_x, q_x) = x$ .

Since  $A'$  is  $\omega$ -r.e. via  $b$ ,

$$p_x = A'(\alpha(1, n_x, q_x)) \dots A'(\alpha(n_x, n_x, q_x))$$

changes at most

$$\begin{aligned} n_x \max\{b(\alpha(m, n_x, q_x)) : 1 \leq m \leq n_x\} &\leq n_x b(\alpha(n_x, n_x, q_x)) \\ &\leq n_x^2 h(|q_x|) \end{aligned}$$

many times. Since  $\tilde{U}(p_x, q_x) = x$  and we can describe  $p_x$  with  $n_x, q_x$  and the number of changes of  $A'(\alpha(1, n_x, q_x)) \dots A'(\alpha(n_x, n_x, q_x))$ , we have

$$\begin{aligned} n_x = C(x | q_x) &\leq 2|n_x| + |n_x^2 h(|q_x|)| + O(1) \\ &\leq 4|n_x| + |h(|q_x|)| + O(1). \end{aligned} \tag{2}$$

To finish, let us prove that for almost all  $x$ ,  $n_x \leq 2|h(|q_x|)| + O(1)$ . Since  $C(x) \leq |q_x| + 2n_x + O(1)$ , this upper bound of  $n_x$  will imply that

$$\begin{aligned} C(x) &\leq |q_x| + h(|q_x|) \\ &= C^A(x) + h(C^A(x)) \end{aligned}$$

for almost all  $x$ , as we wanted. Hence, let us see that  $n_x \leq 2|h(|q_x|)| + O(1)$  for almost all  $x$ . There is a constant  $N$  such that for all  $n \geq N$ ,  $8|n| \leq n$ . We know that for almost all  $x, q_x$  satisfies  $|h(|q_x|)| \geq N$ . Suppose  $x$  has this property. Then either  $n_x \leq |h(|q_x|)|$  or  $4|n_x| \leq n_x/2$ . In the second case  $n_x - 4|n_x| \geq n_x/2$  and by (2),  $n_x/2 \leq |h(|q_x|)| + O(1)$ . So, in both cases, we have  $n_x \leq 2|h(|q_x|)| + O(1)$ .  $\square$

To characterize strong jump traceability, we need a Lemma.

**Lemma 16.** *For all  $x \in \{0, 1\}^*$  and  $d \in \mathbb{N}$ ,*

$$|\{y : C(x, y) \leq C(x) + d\}| = O(d^4 2^d).$$

**Proof.** Chaitin [6] has proved that

$$\forall d, n \in \mathbb{N} |\{\sigma : |\sigma| = n \wedge C(\sigma) \leq C(n) + d\}| = O(2^d).$$

Let  $c$  be such that  $\forall x C(x) \leq str^{-1}(x) + c$ . Consider the partial recursive function  $f(x, y, d)$  which enumerates all strings  $z$  such that  $C(z) \leq str^{-1}(x) + d + c$  until it finds  $z = y$ . If  $z$  was the  $i$ -th string to appear in the enumeration, then  $f(x, y, d)$  is the number  $i$  written in binary with initial zeroes such that  $|f(x, y, d)| = str^{-1}(x) + d + c + 1$ . Notice that it is always possible to write  $f(x, y, d)$  in this way because there are at most  $2^{str^{-1}(x) + d + c + 1}$  such strings  $z$ . If no such  $z$  exists, then  $f(x, y, d) \uparrow$ . Let  $x$  and  $d$  be given. Consider  $y$  such that  $C(x, y) \leq C(x) + d$ . Since  $C(x, y) \leq str^{-1}(x) + d + c$  then  $f(x, y, d) \downarrow$  and

$$\begin{aligned} C(f(x, y, d)) &\leq C(x, y) + 2|d| + O(1) \\ &\leq C(x) + d + 2|d| + O(1) \\ &\leq C(str^{-1}(x) + d + c + 1) + d + 4|d| + O(1). \end{aligned}$$

The last inequality holds because we can compute the string  $x$  from the numbers  $str^{-1}(x) + d + c + 1$  and  $d$ . Let  $n = str^{-1}(x) + d + c + 1$  and  $d' = d + 4|d| + O(1)$ . For fixed  $x$  and  $d$ , the mapping  $y \mapsto f(x, y, d)$  is injective and thus

$$\begin{aligned} |\{y : C(x, y) \leq C(x) + d\}| &\leq |\{\sigma : |\sigma| = n \wedge C(\sigma) \leq C(n) + d'\}| \\ &= O(2^{d'}) = O(d^4 2^d). \end{aligned}$$

This completes the proof. □

**Theorem 17.** *The following are equivalent:*

- (i)  $A$  is strongly jump-traceable;
- (ii) For every order function  $h$  and almost every  $x$ ,  $C(x) \leq C^A(x) + h(C^A(x))$ .

**Proof.** For any function  $f$ , let  $\hat{f}(y) = y + f(y)$  for all  $y$ .

(i) $\Rightarrow$ (ii). Let  $h_0$  be a given order function. It is sufficient to show that  $C(x) \leq \hat{h}(C^A(x)) + O(1)$  for almost all  $x$ , where  $h = \lfloor h_0/2 \rfloor$ . Let  $\alpha$  be a reduction function such that  $J^A(\alpha(x)) = U^A(str(x))$ . Let  $T$  be a trace for  $J^A$  with bound  $g$  such that  $g(\alpha(x)) \leq h(|str(x)|)$ . Let  $m \in \mathbb{N}$  be such that  $U^A(str(m)) = y$  and  $|str(m)| = C^A(y)$ . Since  $y \in T_{\alpha(m)}$ , we can code  $y$  with  $m$  and a number not greater than  $g(\alpha(m))$  (representing the place ( $\leq g(\alpha(m))$ ) within the enumeration of  $T_{\alpha(m)}$  at which  $y$  is enumerated), using at most

$$|str(m)| + g(\alpha(m)) \leq C^A(y) + h(C^A(y))$$

many bits. Then  $\forall y C(y) \leq \hat{h}(C^A(y)) + O(1)$ .

(ii) $\Rightarrow$ (i). Since there are at most  $2^n - 1$  programs of length  $< n$ ,  $\forall n \exists x [|x| = n \wedge n \leq C(x)]$ . Let  $c$  be a constant such that

$$\forall x [J^A(|x|) \downarrow \Rightarrow C^A(x, J^A(|x|)) \leq |x| + c].$$

This last inequality holds because, given  $x$ , we can compute  $J^A(|x|)$  relative to  $A$ .

Let  $h$  be any order function and let us prove that  $A$  is jump-traceable via  $h$ . Define the order function  $g$  such that for almost all  $e$ ,  $3^{g(e+c)} \leq h(e)$ . By hypothesis, for almost all  $x$ , if  $J^A(|x|) \downarrow$  then

$$\begin{aligned} C(x, J^A(|x|)) &\leq \hat{g}(C^A(x, J^A(|x|))) \\ &\leq |x| + g(|x| + c) + c. \end{aligned}$$

Define the trace

$$T_e = \{y : \forall x [ |x| = e \Rightarrow C(x, y) \leq e + g(e + c) + c ]\}.$$

It is clear that for almost all  $e$ , if  $J^A(e) \downarrow$  then  $J^A(e) \in T_e$ , because given  $x$  such that  $|x| = e$ , we have  $C(x, J^A(e)) \leq e + g(e + c) + c$ . To verify that for almost all  $e$ ,  $|T_e| \leq h(e)$ , suppose  $y \in T_e$ . Take  $x$ ,  $|x| = e$  and  $C(x) \geq e$ . Then

$$\begin{aligned} C(x, y) &\leq e + g(e + c) + c \\ &\leq C(x) + g(e + c) + c. \end{aligned}$$

By Lemma 16, for almost all  $e$  there are at most  $3^{g(e+c)} \leq h(e)$  such  $y$ 's in  $T_e$ .  $\square$

In [20], it was proven that there is a super-low set which is not jump-traceable (namely, a super-low Martin-Löf random set). In contrast, from Theorem 15 and Theorem 17 we can conclude that the strong version of super-lowness implies strong jump-traceability.

**Corollary 18.** *If  $A'$  is well-approximable then  $A$  is strongly jump-traceable.*

## 6 Variations on $K$ -triviality

Throughout this section, let  $p : \mathbb{N} \rightarrow \mathbb{N}$  be strictly increasing such that in addition  $\lim_n p(n) - n = \infty$ . We call  $p$  an *estimation function* if, in addition,  $p(n) = \lim_s p_s(n)$  where  $p_{s+1}(n) \leq p_s(n)$ , and the function  $\lambda s, n. p_s(n)$  is recursive. An example of such a function is  $q(n) = n + 5 \cdot \min\{K(m) : m \geq n\}$  with the approximation  $q_s(n) = n + 5 \cdot \min\{K_s(m) : s \geq m \geq n\}$ .

Recall that  $A$  is  $K$ -trivial iff

$$\exists c \forall n K(A \upharpoonright n) \leq K(n) + c.$$

Nies [19] has shown that  $A$  is  $K$ -trivial if and only if  $A$  is low for  $K$ , that is,  $\exists c \forall x K(x) \leq K^A(x) + c$ . In this section we weaken the notion of lowness for  $K$ :

**Definition 19.** (i) A set  $A$  is *weakly  $p$ -low* iff  $\forall n K(A \upharpoonright n) \leq p(K(n) + c_0) + c_1$  for some constants  $c_0$  and  $c_1$ . Let  $\mathcal{K}[p]$  denote the class of such sets.

(ii) A set  $A$  is  *$p$ -low* iff  $\forall y K(y) \leq p(K^A(y) + c_0) + c_1$  for some constants  $c_0$  and  $c_1$ . Let  $\mathcal{M}[p]$  denote the class of such sets.

**Proposition 20.** (i) If  $A \in \mathcal{M}[p]$  and  $B \leq_T A$ , then  $B \in \mathcal{M}[p]$ .

(ii) If  $A \in \mathcal{K}[p]$  and either  $B \leq_K A$  or  $B \leq_{\text{wtt}} A$ , then  $B \in \mathcal{K}[p]$ .

(iii) Suppose  $p$  is an estimation function. Then no random set is in  $\mathcal{K}[p]$ .

(iv) If  $A, B \in \mathcal{K}[p]$  and  $A, B$  are r.e., then

$$A \oplus B = \{2x : x \in A\} \cup \{2x + 1 : x \in B\} \in \mathcal{K}[p].$$

(v)  $\mathcal{M}[p] \subseteq \mathcal{K}[p]$ .

**Proof.** (i). Since  $B \leq_T A$ , there exists a constant  $c_2$  such that for each string  $y$ ,  $K^A(y) \leq K^B(y) + c_2$ . Then

$$\begin{aligned} K(y) &\leq p(K^A(y) + c_0) + c_1 \\ &\leq p(K^B(y) + c_0 + c_2) + c_1. \end{aligned}$$

(ii). This is trivial for  $\leq_K$ . Now suppose  $B = \Gamma^A$  for a weak truth-table reduction  $\Gamma$  with recursive bound  $f$ . Without loss of generality, we may assume  $f$  strictly increasing. Given  $A \upharpoonright f(n)$  we can compute  $n$  and  $B \upharpoonright n$ , and then there is a constant  $c_2$  such that for all  $n$ ,

$$\begin{aligned} K(B \upharpoonright n) &\leq K(A \upharpoonright f(n)) + c_2 \\ &\leq p(K(f(n)) + c_0) + c_1 + c_2. \end{aligned}$$

Since  $f$  is recursive, we have  $K(f(n)) \leq K(n) + O(1)$  and hence  $B \in \mathcal{K}[p]$ .

(iii). Assume  $\forall n K(A \upharpoonright n) > n - c$  and  $A \in \mathcal{K}[p]$  via constants  $c_0$  and  $c_1$ . Since  $p$  is an estimation function,  $p(n) = \lim_s p_s(n)$  where  $p_{s+1}(n) \leq p_s(n)$ , and the function  $\lambda s, n. p_s(n)$  is recursive. Define the strictly increasing recursive function  $\tilde{p}(0) = p_0(0)$  and  $\tilde{p}(k+1) = p_0(j)$ , where  $j = \min\{i : i > k \wedge p_0(i) > \tilde{p}(k)\}$ . Since  $\tilde{p} \geq p$ ,  $A \in \mathcal{K}[\tilde{p}]$ . Define the Kraft-Chaitin set  $\{\langle i, n_i \rangle : i \in \mathbb{N}^+ \wedge n_i = \tilde{p}(i + d + c_0) + c_1 + c\}$  for  $M_d$  with  $d$  given in advance by the Recursion Theorem. Then  $K(n_i) \leq i + d$  and hence  $\tilde{p}(K(n_i) + c_0) \leq \tilde{p}(i + d + c_0)$ . Finally,

$$\begin{aligned} K(A \upharpoonright n_i) &\leq \tilde{p}(K(n_i) + c_0) + c_1 \\ &\leq \tilde{p}(i + d + c_0) + c_1 = n_i - c \end{aligned}$$

and this is a contradiction.

(iv). Ignoring constants, for each  $n$ ,

$$\begin{aligned} K(A \oplus B \upharpoonright n) &\leq K(A \oplus B \upharpoonright 2n) \\ &\leq \max\{K(A \upharpoonright n), K(B \upharpoonright n)\} \\ &\leq p(K(n)). \end{aligned}$$

In the second inequality we used [13, Theorem 6.4].

(v). Again ignoring constants, for all  $n$ ,

$$\begin{aligned} K(A \upharpoonright n) &\leq p(K^A(A \upharpoonright n)) \\ &\leq p(K^A(n)) \\ &\leq p(K(n)). \end{aligned}$$

This completes the proof.  $\square$

The following proposition shows a connection between jump-traceability and  $p$ -lowness. In Theorem 17 we proved a similar result, relating strong jump-traceability and plain Kolmogorov complexity.

**Proposition 21.** (i) *Suppose  $p$  is a recursive function. There is a constant  $c$  such that if  $A \in \mathcal{M}[p]$  via constants  $c_0$  and  $c_1$  then  $A$  is jump-traceable via  $h(x) = 2^{p(2|x|+c_0+c)+c_1+1}$ ,*

(ii) *There is a reduction function  $\alpha$  such that if  $A$  is jump-traceable via  $h$  then  $A \in \mathcal{M}[p]$  for  $p(z) = 3z + 2|h(\alpha(2^{z+1}))|$ .*

**Proof.** For (i), we know that there is a constant  $c$  such that  $K^A(J^A(x)) \leq 2|x| + c$  because we can compute  $J^A(x)$  from  $x$  and the oracle  $A$ . Define the trace

$$T_x = \{U(\sigma) : |\sigma| \leq p(2|x| + c_0 + c) + c_1\}.$$

Clearly  $|T_x| \leq 2^{p(2|x|+c_0+c)+c_1+1}$ . Let  $y = J^A(x)$ . By hypothesis  $K(y) \leq p(K^A(y) + c_0) + c_1$  and then  $K(y) \leq p(2|x| + c + c_0) + c_1$ . Hence  $y \in T_x$ .

For (ii), let  $\alpha$  be a reduction function such that  $J^A(\alpha(x)) = U^A(\text{str}(x))$ . Let  $T$  be a trace for  $J^A$  with bound  $h$  and let us define the trace

$$\tilde{T}_n = \bigcup_{x:|\text{str}(x)|=n} T_{\alpha(x)}.$$

Notice that

$$\begin{aligned} |\tilde{T}_n| &\leq \sum_{x:|\text{str}(x)|=n} h(\alpha(x)) \\ &\leq 2^n h(\alpha(2^{n+1})), \end{aligned}$$

since  $\alpha$  is increasing. Let  $m \in \mathbb{N}$  be such that  $U^A(\text{str}(m)) = y$  and  $|\text{str}(m)| = K^A(y)$ . Since  $y \in T_{\alpha(m)}$ , we know that  $y \in \tilde{T}_{|\text{str}(m)|}$ , hence we describe  $y$  by saying “ $y$  is the  $i$ -th element enumerated into  $\tilde{T}_{|\text{str}(m)|}$ ”. If we code  $|\text{str}(m)|$  in unary and we code  $i$  with

$$\begin{aligned} 2|i| &\leq 2|2^{|\text{str}(m)}| h(\alpha(2^{|\text{str}(m)}+1})) \\ &\leq 2|\text{str}(m)| + 2|h(\alpha(2^{|\text{str}(m)}+1}))| \end{aligned}$$

many bits, we have  $K(y) \leq p(K^A(y)) + O(1)$ , for  $p(z) = 3z + 2|h(\alpha(2^{z+1}))|$ .  $\square$

**Corollary 22.** *A is jump-traceable iff there exists a recursive function  $p$  (of the type considered in this section) such that  $A \in \mathcal{M}[p]$ .*

Figueira, Stephan and Wu [15, Proposition 6] used a universal machine which has the property that there is an approximation  $K_s$  of  $K$  from above with  $K_x(x) = K(x)$  for all  $x \in X$  where  $X = \{x : \forall y > x (K(y) > K(x))\}$ . For the following Theorem, such a universal machine is assumed. The example shows that there is a set in  $\mathcal{M}[q]$  where  $q$  is as defined at the beginning of Section 6 which is not  $K$ -trivial. Note that  $r$  differs from the function in Lemma 6 only by using  $K$  instead of  $C$  and has the same properties as the function given there. In particular, for each order function  $h$  we have  $r(n) \leq n + h(n)$  for almost each  $n$ , and thus the set constructed satisfies the analog for  $K$  of the condition in Theorem 17 characterizing strong jump traceability. In contrast to this result, Cholak, Downey and Greenberg [9] have shown that each strongly jump-traceable set is in  $\Delta_2^0$ .

**Theorem 23.** *Let  $r(n) = \min\{K(m) : m \geq n\}$  and  $q(n) = n + 5 \cdot r(n)$ . Then there is a set  $A \in \mathcal{M}[q] \setminus \Delta_2^0$ .*

**Proof.** Note that the set  $X = \{x : \forall y > x \forall t (K_t(y) > K_x(x))\}$  is co-r.e. and that it has a co-r.e. subset  $Y$  of the form  $\{y_0, y_1, \dots\}$  such that, for all  $n$ ,  $y_n = K(y_{n+1}) = K_{y_{n+1}}(y_{n+1})$ . As  $K(0) > 0$  one might have the undesirable property that  $y_{n+1} < y_n$  for some  $n$ . But as there are only finitely many numbers  $x$  with  $K(x) > x$ , one simply adds to the construction of  $Y$  the condition that  $y_0$  is taken to be the first element of  $X$  larger than these finitely many exceptions and so one has the additional property that  $y_{n+1} > y_n$  for all  $n$ .

Now one defines a partition  $I_0, I_1, \dots$  of the natural numbers into intervals such that  $|I_x| = K_x(K_x(x))$  and  $\max(I_x) + 1 = \min(I_{x+1})$ . Note that none of these intervals is empty as  $K_x(K_x(x)) > 0$  for all  $x$  which is due to the fact that a prefix-free universal machine is undefined on the empty input.

Having the partition, one defines a partial-recursive function  $\psi$  in stages  $s$  where one does the following algorithm where  $\psi$  is everywhere undefined before stage 0. The set  $A$  will be chosen such that its characteristic function is a suitable extension of  $\psi$ . Let  $\psi_s$  denote the approximation to  $\psi$  before stage  $s$ .

- Find the least  $x, y$  such that  $x \leq s$ ,  $y \in I_x$ ,  $\psi_s(y)$  is undefined and either (1)  $x \notin Y_s$  or (2) there is a string  $\sigma \in \{0, 1\}^{\max(I_x)+1}$  such that  $K_s(\sigma) < K_s(x) + 0.5 \cdot \log(|I_x|)$  and  $\sigma$  is consistent with  $\psi_s$ , that is,  $\psi_s(z) = \sigma(z)$  for all  $z \in \text{domain}(\psi_s) \cap \{0, 1, \dots, \max(I_x)\}$ .
- In the case that no  $x, y$  were found, let  $\psi_{s+1} = \psi_s$ .
- In the case that  $x, y$  were found according to condition (1), let  $\psi_{s+1}(y) = 0$  and let  $\psi_{s+1}(z) = \psi_s(z)$  for all  $z \neq y$ .
- In the case that  $x, y$  were found according to condition (2), let  $\psi_{s+1}(y) = 1 - \sigma(y)$  and let  $\psi_{s+1}(z) = \psi_s(z)$  for all  $z \neq y$ .

Now let  $A$  be a set whose characteristic function extends  $\psi$  and which is low for  $\Omega$ . Such a set  $A$  exists since  $\psi$  defines a  $\Pi_1^0$  class and Downey, Hirschfeldt, Miller and Nies [12] showed every  $\Pi_1^0$  class (of sets) has a member which is low for  $\Omega$ .

Reviewing the construction of  $\psi$ , condition (1) enforces that  $\psi$  is defined on the complete interval  $I_x$  if  $x \notin Y$  and condition (2) enforces that if  $x = y_n$  and  $n$  is large enough then the Kolmogorov complexity of  $A \upharpoonright \max(I_{y_n})$  is at least  $K(y_n) + \log(|I_{y_n}|)/2$ . To see this, one should have in mind that  $x \rightarrow \max(I_x)$  is a recursive injective function, that  $K_{y_n}(y_n) = K(y_n)$  and that the number of  $\sigma$  of length  $\max(I_{y_n}) + 1$  with  $K(\sigma) \leq K(y_n) + \log(|I_{y_n}|)/2$  is bounded by a function proportional to  $\sqrt{|I_{y_n}|}$ . So there will for all sufficiently large  $n$  remain some elements in  $I_{y_n}$  where  $\psi$  is undefined. As the intervals  $I_{y_n}$  are of unbounded length, this enforces that for sufficiently large  $n$  the value of  $K(A \upharpoonright \max(I_{y_n}))$  is at least  $K(y_n) + \log(|I_{y_n}|)/2$  while on the other hand  $K(\max(I_{y_n}))$  is only a constant above  $K(y_n)$ . So  $A$  is not  $K$ -trivial. Since every low for  $\Omega$  set is either  $K$ -trivial or not  $\Delta_2^0$ ,  $A$  is also not  $\Delta_2^0$ , that is, not limit-recursive.

Now it is shown that the set  $A$  constructed satisfies  $K^A(x) \leq q(K(x)) + c_0$  for some constant  $c_0$  and all  $x$ . This needs some facts about the sequence  $y_0, y_1, \dots$  and the complexities of these strings relative to  $A$ .

For ease of notation,  $U^A$  denotes the universal prefix-free machine relative to  $A$  and  $U = U^\emptyset$  the unrelativized one. Let  $a_n$  be an input of shortest length such that  $U^A(a_n) = y_n$  and let  $b_n$  be an input of length  $y_{n-1}$  such that  $U(b_n) = y_n$ .

Now consider all the  $n$  such that  $|a_n| \leq y_{n-1} - 2y_{n-2}$ . Then one has a prefix-free machine  $V^A$  and a partial-recursive coding function  $\theta$  such that

- $V^A(b_{n-1}a_n)$  computes  $\Omega_{y_n} \upharpoonright y_{n-1} - y_{n-2} - c_1$ ;
- $U(\theta(b_{n-1}\Omega \upharpoonright y_{n-1} - y_{n-2} - c_1))$  computes  $\min\{s : \Omega_s \upharpoonright (y_{n-1} - y_{n-2} - c_1) = \Omega \upharpoonright (y_{n-1} - y_{n-2} - c_1)\}$ .

where the constant  $c_1$  is so large that  $\theta$  can be chosen such that  $|\theta(b_{n-1}d)| \leq y_{n-1}$  for all  $d \in \{0, 1\}^{y_{n-1} - y_{n-2} - c_1}$ . As a consequence, the computation  $U(\theta(b_{n-1}\Omega \upharpoonright y_{n-1} - y_{n-2} - c_1))$  needs less than  $y_n$  steps. Thus,  $V^A(b_{n-1}a_n)$  computes  $\Omega \upharpoonright y_{n-1} - y_{n-2} - c_1$  and  $|b_{n-1}a_n| = y_{n-2} + |a_n| \leq y_{n-1} - y_{n-2}$ . Since  $\Omega$  is random relative to  $A$ , this can happen only for finitely many  $n$  and one has that  $|a_n| > y_{n-1} - 2y_{n-2}$  for almost all  $n$ .

Now assume that  $n > 1$  and  $|a_n| > y_{n-1} - 2y_{n-2}$ . Let  $E_n = \{e : U^A(e) \text{ needs at least } \min(I_{y_n}) \text{ and at most } \min(I_{y_{n+1}}) - 1 \text{ steps}\}$ . Note that for  $e \in E_n$ ,  $b_n$  is that string of length  $y_{n-1}$  for which  $U(b_n)$  terminates last within the computation-time of  $U^A(e)$  and  $y_n = U(b_n)$ . So one has a constant  $c_2$  and for each  $e$  a prefix-free input  $d$  of length  $|e| + K(y_{n-1}) + c_2$  such that  $U^A(d) = y_n$ . This gives that there is a constant  $c_3$  with

$$\sum_{e \in E_n} 2^{-|e| - c_2 - K(y_{n-1})} < 2^{c_3 - |a_n|}$$

what using  $|a_n| > y_{n-1} - 2y_{n-2}$  can be transformed to

$$\sum_{e \in E_n} 2^{y_{n-1} - c_2 - c_3 - 3y_{n-2} - e} < 1.$$

There is a partial-recursive function  $g$  such that  $g(b_n) = |I_{y_0} \cup I_{y_1} \cup \dots \cup I_{y_n}|$ . Now one can construct a prefix-free machine which on input  $bd$  with  $U(b)$  being defined and  $|d| = g(b_n)$  enumerates requests of weight at most  $2^{-b-d}$  with the additional constraint that, in the case that  $b = b_n$  and  $d$  is the restriction of  $A$  to  $I_{y_0} \cup I_{y_1} \cup \dots \cup I_{y_n}$ , the requests are just an enumeration of the set

$$\{\langle |b_n| + g(b_n) + |e| + c_2 + c_3 + 3y_{n-2} - y_{n-1}, U^A(e) \rangle : e \in E_n\}.$$

Recall that the weight of a request  $\langle i, j \rangle$  is  $2^{-i}$ . So the sum of the weights of all requests is at most 1. Note from  $b_n$  and  $d$  one can compute  $y_0, y_1, \dots, y_n$  and  $A$  on  $I_{y_0} \cup I_{y_1} \cup \dots \cup I_{y_n}$  so that the enumeration is effective. By the inequality

$$\sum_{e \in E_n} 2^{y_{n-1} - c_2 - c_3 - 3y_{n-2} - e} < 1.$$

from above one has that the bound on the weight of the requests is kept. Assume that  $|e| = K^A(x)$  and  $U^A(e) = x$  and  $x$  is so large that  $e \in E_n$  for an  $n$  satisfying that  $g(b_n) \leq 2y_{n-2}$  and that  $n$  does not fall under the finitely many exceptions considered above. Then there is a request of the form  $\langle |e| + g(b_n) + c_2 + c_3 + 3y_{n-2}, x \rangle$ . It follows from the Kraft-Chaitin Theorem that there is a constant  $c_4$  with  $K^A(x) \leq |e| + 5y_{n-2} + c_4$  for the  $n$  with  $e \in E_n$ .

As for almost all  $n$ ,  $|a_n| > y_{n-1} - 2y_{n-2}$  and as one can compute  $y_n$  relative to  $A$  from  $y_{n-2}$  plus an upper bound on  $y_n$ , one has that for almost all  $n$  and every  $e$  with  $U^A(e)$  needing more than  $y_n$  steps that  $|e| > y_{n-1} - 3y_{n-2} - c_5$  for some constant  $c_5$ . Since  $r$  grows slower than every unbounded and nondecreasing recursive function and  $y_{n-1} - 3y_{n-2} - c_5 > y_{n-1}/2$  for almost all  $n$ , there is a constant  $c_6$  such that  $r(e) \geq r(y_n) - c_6 = y_{n-2} - c_6$  where  $c_6$  is independent of  $e, n$  as long as  $e \in E_n$ . So one has that  $K(U^A(e)) \leq |e| + 5r(|e|) + c_4 + 5c_6$ .

One can now cover the case the  $x = U^A(e)$  the finitely many  $x$  where  $U^A(e)$  needs at most  $\min(I_{y_{n+1}}) - 1$  steps for some of the finitely many exceptional  $n$  in the case distinction above by taking  $c_0$  to be sufficiently much larger than  $c_4 + 5c_6$  and obtains that

$$\forall x \ K(x) \leq K^A(x) + 5r(K^A(x)) + c_0 = q(K^A(x)) + c_0$$

what completes the proof. □

One should note that the real difficulty of this construction stems from the fact that the constructed set has to be  $p$ -low and not only weakly  $p$ -low. For estimation functions, the construction of weakly  $p$ -low sets is quite straight-forward. Note that the resulting set is not  $K$ -trivial as it is Turing complete.

**Proposition 24.** *Let  $p$  be an estimation function. Then there is a Turing complete r.e. set  $A$  which is weakly  $p$ -low and also satisfies the corresponding property for  $C$ : there are constants  $c_K, c_C$  such that  $K(A \upharpoonright x) \leq p(K(x)) + c_K$  and  $C(A \upharpoonright x) \leq p(C(x)) + c_C$  for all  $x$ .*

**Proof.** For defining an enumeration of  $A$ , fix a one-one enumeration  $b_0, b_1, \dots$  of the halting problem and approximations  $C_s, K_s$  to  $C, K$ . Let  $A_0 = \emptyset$ . At stage  $s + 1$ , let  $a_m$  be the  $m$ -th nonelement of  $A_s$  in ascending order. Now the set  $A_{s+1}$  is computed as follows.

- Let  $n$  be the minimum of all  $m$  such that one of the following conditions holds:
  - $a_m > s$ ;
  - $b_s \leq m$ ;
  - $p_s(K_s(k)) - K_s(k) \leq m$  for some  $k$  with  $a_m \leq k \leq s$ ;
  - $p_s(C_s(k)) - C_s(k) \leq m$  for some  $k$  with  $a_m \leq k \leq s$ .
- Let  $A_{s+1} = A_s \cup \{x : a_n \leq x \leq s\}$ .

This set  $A$  satisfies the following properties:

- $A$  is coinfinite and r.e.;
- $A$  is Turing complete;
- $K(A \upharpoonright x) \leq p(K(x)) + c_K$  for some constant  $c_K$  and all  $x$ ;
- $C(A \upharpoonright x) \leq p(C(x)) + c_C$  for some constant  $c_C$  and all  $x$ .

The first property states the obvious fact that  $A$  is r.e. by construction. The other fact that  $A$  is co-infinite needs some more thought. Assume by way of contradiction that  $|\overline{A}| = m$  for some finite number  $m$ . Let  $a_0, a_1, \dots, a_{m-1}$  denote the nonelements of  $A$  in ascending order and assume that  $s$  is so large that the following conditions hold:

- if  $b_t \leq m$  then  $t < s$ ;
- for all  $x \in A - A_s$  there is no  $k \geq x$  and no  $e \geq \min\{C(k), K(k)\}$  such that  $p(e) - e \leq m$ ;
- if  $x \leq a_{m-1} + 1$  then  $x \in A \Leftrightarrow x \in A_s$ .

Then one can see that the parameters  $a_0, a_1, \dots, a_{m-1}$  chosen in the definition of step  $s$  coincide with the  $m$  least nonelements of  $A$  and are just not enumerated. Furthermore,  $a_m$  is also defined as the next nonelement of  $A_s$ . Note that  $a_m \leq s$  as  $s \notin A_s$ . Now one can see that  $a_m$  is not enumerated into  $A_{s+1}$  because the  $n$  selected is larger than  $m$ : for all  $m' < m$ ,  $n \neq m'$  because otherwise  $a_0, a_1, \dots, a_{m-1}$  would not remain outside  $A$ ; furthermore,  $n \neq m$  as the first and second item in the conditions on  $s$  together with the facts that  $p_s$  approximates  $p$  from above and  $a_m \leq s$  imply that  $m$  does not satisfy the search-conditions. So  $a_m \notin A_{s+1}$  and one can show by induction that  $a_m \notin A_t$  for all  $t > s$ , this contradicts the assumption that  $|\overline{A}| = m$ . Therefore,  $A$  is coinfinite.

The second property follows from the construction. If  $a_0, a_1, \dots$  are the nonelements of  $A$  in ascending order, then  $b_s \leq m$  implies  $s \leq a_m$ . Thus  $m$  is in the halting problem iff  $m \in \{b_0, b_1, \dots, b_{a_m}\}$  and so the halting problem is Turing reducible to  $A$ .

The third property can be seen as follows: Given  $x$  and the shortest description  $\sigma$  for  $x$  with respect to a fixed prefix-free universal machine, let  $n$  be the number of nonelements of  $A$  below  $x$ . Then one can construct a prefix-free machine which from input  $1^n 0 \sigma$  first evaluates the universal machine on  $\sigma$  to get the value  $x$  and then searches for a stage  $s$  such that  $A_s$  contains all but  $n$  elements below  $x$ . Having this  $x$  and  $s$ , the machine outputs  $A_s \upharpoonright x$ . If  $\sigma$  and  $n$  are chosen correctly, then the output is correct. Thus one has that  $K(A \upharpoonright x)$  is at most  $K(x) + n + c_K$  where the constant  $c_K$  comes from translating the given prefix-free coding of  $K(A \upharpoonright x)$  of length  $K(x) + n + 1$  for some machine into inputs for the universal machine. Furthermore, for all sufficiently large  $s$ ,  $K_s(x) + n \leq p_s(K_s(x))$  as otherwise the marker  $a_{n-1}$  would move. Therefore  $K(x) + n \leq p(K(x))$  and  $A$  is weakly  $p$ -low.

The fourth property can be proven analogously; here the constructed machine is not prefix-free and  $\sigma$  is the shortest input producing  $x$  with respect to some fixed universal plain machine, nevertheless  $\sigma$  and  $n$  can of course still be recovered from  $1^n 0 \sigma$ . The rest of the proof follows the previous item but is working with  $C$  in place of  $K$ . This completes the proof of the whole result.  $\square$

For any estimation function  $p$  and the above constructed  $A \in \mathcal{K}[p]$ ,  $\Omega \leq_T A$  and thus  $A \notin \mathcal{M}[p]$  by Proposition 20 (i) and (iii). Thus the inclusion from Proposition 20 (v) is strict.

**Corollary 25.** *For all estimation functions  $p$ ,  $\mathcal{M}[p] \subset \mathcal{K}[p]$ .*

**Proposition 26.** *For every estimation function  $p$  there is a whole Turing degree outside  $\Delta_2^0$  contained in  $\mathcal{K}[p]$ .*

**Proof.** For any estimation function  $p$  one can consider the estimation function  $q$  given as  $q(n) = n + \log(p(n) - n)/2$ . Then one can construct a r.e. set  $A$  as in Proposition 24 which is in  $\mathcal{K}[q]$ .

The set  $A$  is not recursive. Thus, due to Yates' version of the Friedberg-Muchnik Splitting Theorem [21, Theorem IX.2.4 and Exercise IX.2.5], one can construct a partial-recursive  $\{0, 1\}$ -valued function  $\psi$  with domain  $A$  such that  $\psi^{-1}(0), \psi^{-1}(1)$  form a recursively inseparable pair, that is,  $\psi$  does not have a total extension. Actually, given a one-one enumeration  $a_0, a_1, \dots$  of  $A$ , this function  $\psi$  can be inductively defined on this domain by taking  $\psi(a_s)$  in  $\{0, 1\}$  such that  $\psi(a_s)$  differs from  $\varphi_{e,s}(a_s)$  for the least  $e$  where either  $e = s$  or  $\varphi_{e,s}(a_s)$  is defined and  $\psi(a_t) = \varphi_{e,s}(a_t)$  for all  $t < s$  with  $a_t \in \text{domain}(\varphi_{e,s})$ .

Every total extension  $B$  of  $\psi$  is in  $\mathcal{K}[p]$  as given any  $n$  and any  $x$ , the number  $m$  of places below  $x$  where  $\psi$  is undefined satisfies  $m < q(K(x)) - K(x)$ . Let  $x_1, x_2, \dots, x_m$  be these places. Let  $\sigma$  be the shortest input such that the universal machine for  $K$  computes  $x$ . Then one can code  $B \upharpoonright x$  by  $1^m 0 B(x_1) B(x_2) \dots B(x_m) \sigma$  and thus has that  $K(B \upharpoonright x)$  is below  $p(K(x))$ . As one can take  $B$  to have hyperimmune-free Turing degree [21, Theorem V.5.34] and as  $\mathcal{K}[p]$  is closed under wtt-reducibility, one has that a whole Turing degree outside  $\Delta_2^0$  is contained in  $\mathcal{K}[p]$ .  $\square$

Note that the above result also holds with  $C$  in place of  $K$ , the proof is exactly the same. So given an estimation function  $p$ , one can construct a hyperimmune-free Turing degree only consisting of sets  $E$  satisfying  $C(E \upharpoonright x) \leq p(E(x))$  for all  $x$  up to an additive constant. Unfortunately, it is not guaranteed that this degree is also strongly jump-traceable, it is even a bit unlikely, as only the use of total  $E$ -recursive functions but not of the jump is recursively bounded in the case of a set  $E$  of hyperimmune-free degree.

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